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1 Straight Lines and Linear Equations

Let us first recall that in the $xy$–plan, any point $P$ corresponds to a pair of numbers $(x, y)$ as indicated in the following diagram, where the number $x$ is called the $x$–coordinate of the point $P$, and $y$ is called the $y$–coordinate of $P$, and $(x, y)$ is simply called the coordinates of $P$.

If we have an equation, such as $2x - y - 3 = 0$, then we can find a graph for it in the $xy$–plane, the graph consists of all the points whose coordinates $(x, y)$ satisfy the equation.

Thus, for the equation $2x - y - 3 = 0$, we can solve for $y$ in terms of $x$, and obtain

$$y = 2x - 3.$$ 

It follows that, for any value of $x$, $(x, y) = (x, 2x - 3)$ is a point on the graph of the equation. For example, if we take $x = -2, -1, 0, 1, 2, 3$, then we obtain the points $(-2, -7), (-1, -5), (0, -3), (1, -1), (2, 1), (3, 3)$. This is best expressed by the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 2x - 3$</td>
<td>-7</td>
<td>-5</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>...</td>
</tr>
</tbody>
</table>

Drawing the exact graph of an equation in two variables is usually an impossible task because it would involve plotting infinitely many points. For example, the graph of $2x - y - 3 = 0$ contains all the points $(x, 2x - 3)$ where $x$ runs through all the real numbers. In general practice, one chooses many points satisfying the given equation and plots only these points and then join them by a smooth curve. If these points are chosen properly, the obtained curve exhibits the general nature of the graph.
Going back to the equation $2x - y - 3 = 0$, we plot the points chosen above, and obtain a straight line. We say $2x - y - z = 0$ is the equation of this line.

In general, a straight line has an equation of the form

$$Ax + By + C = 0$$

where $A, B, C$ are constants and $A, B$ are not both zero.

An equation of this form is called a **linear equation**. Therefore, there is a one to one correspondence between straight lines and linear equations (of the variables $x$ and $y$).

Consider again the straight line $2x - y - 3 = 0$. We know $(1, -1)$ and $(3, 3)$ are two points on it. Denote the point $(1, -1)$ by $P$, and $(3, 3)$ by $Q$, and let $R$ be the point as indicated in the diagram. The difference between the $x$–coordinates of $P$ and $Q$ equals the length of $PR$, and is called the **run** from $P$ to $Q$:

$$\text{run} = PR = 3 - 1 = 2.$$  

The difference between the $y$–coordinates of $P$ and $Q$ equals the length of $QR$, and is called the **rise** from $P$ to $Q$:

$$\text{rise} = QR = 3 - (-1) = 4$$

Clearly,

$$\frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2.$$  

It turns out that, the ratio $\frac{\text{rise}}{\text{run}}$ does not depend on our particular choice of $P$ and $Q$, i.e., for any two points $P'$ and $Q'$ on the line, say $(x_1, y_1)$ and $(x_2, y_2)$,

$$\frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2x_2 - 3) - (2x_1 - 3)}{x_2 - x_1} = \frac{2(x_2 - x_1)}{x_2 - x_1} = 2.$$
We call this common ratio the **slope** of the straight line. For a general straight line, let $P$ and $Q$ be any two points on it, with coordinates $(x_1, y_1)$ and $(x_2, y_2)$, respectively. We define the **run from $P$ to $Q$** as $x_2 - x_1$ and the **rise from $P$ to $Q$** as $y_2 - y_1$. The slope of the line is defined to be the ratio of rise to run. It is usually denoted by the letter $m$. Hence

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

As in the case for the particular straight line $2x - y - z = 0$, it is easily proved that the slope defined in this way does not depend on the particular choice of $P$ and $Q$. However, if the line is vertical, we necessarily have $x_1 = x_2$, and thus $\frac{y_2 - y_1}{x_2 - x_1}$ is not defined. We agree that **slope is not defined for vertical lines**.

**Example 1.** Find the slope of the straight line given by the equation $y = ax + b$, where $a$ and $b$ are constants.

**Solution.** Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points on the line. Then, $y_1 = ax_1 + b$, $y_2 = ax_2 + b$, and

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(ax_2 + b) - (ax_1 + b)}{x_2 - x_1} = \frac{ax_2 - ax_1}{x_2 - x_1} = a.$$

This example shows that given the equation of a straight line, it is easy to find the slope of the line. For example, from $2x - y - 3 = 0$ we obtain $y = 2x - 3$ and hence the slope is 2.

**Example 2.** Find the equation of the straight line passing through a given point $(x_1, y_1)$ with slope $m$.

**Solution** From example 1 we know that any line with equation $y = mx + b$ has slope $m$. We use $(x_1, y_1)$ to determine the constant $b$ as follows.

$$y_1 = mx_1 + b.$$

Hence $b = y_1 - mx_1$.

Therefore, $y = mx + b = mx + y_1 - mx_1$,

$$\text{i.e. } y = y_1 + m(x - x_1) \quad \text{or } y - y_1 = m(x - x_1)$$
This is called the **point-slope formula** for the line.

**Example 3.** Find the equation of the straight line passing through \((1, -2)\) and \((5, 6)\).

**Solution** \[ m = \frac{6 - (-2)}{5 - 1} = \frac{8}{4} = 2. \]

Hence, the equation of the straight line is (using \((x_1, y_1) = (1, -2)\))

\[
\begin{align*}
  y - (-2) &= 2(x - 1) = 2x - 2 \\
  \text{or} \quad y &= 2x - 4 \\
\end{align*}
\]

**Remark** If we use \((x_1, y_1) = (5, 6)\) in example 3, we obtain

\[
\begin{align*}
  y - 6 &= 2(x - 5) = 2x - 10 \\
\end{align*}
\]

and arrive at the same equation \(y = 2x - 4\).

**Example 4** Given the linear equation \(2x + 3y + 6 = 0\), find the slope of its graph and sketch the straight line.

**Solution** Solving for \(y\) from the equation we obtain

\[
y = -\frac{2}{3}x - 2
\]

Therefore, \(m = -\frac{2}{3}\).

To sketch the graph, we just need two points on the line. Let \(x = 0\) in the equation \(2x + 3y + 6 = 0\), we get

\[
3y + 6 = 0, \quad y = -2. \quad \text{(We can also use } y = -\frac{2}{3}x - 2 \text{ here)}
\]

Thus \((0, -2)\) is on the line.

Let \(y = 0\) in the equation, we obtain

\[
2x + 6 = 0, \quad x = -3
\]

Thus \((-3, 0)\) is also on the line.
Clearly $(0, -2)$ is the point where the line intersects with the $y$–axis, and $(-3, 0)$ is the intersection of the line with the $x$–axis.

Remarks

(1) The equation $y = b$ gives a horizontal line which intersects the $y$–axis at $(0, b)$ and never intersects the $x$–axis unless $b = 0$. Similarly, $x = a$ gives a vertical line intersecting the $x$–axis at $(a, 0)$. Except these two kinds of lines, all other lines intersect both the $x$-axis and $y$–axis. If a straight line intersects the $x$–axis at $(a, 0)$ and the $y$–axis at $(0, b)$, then $a$ is called its $x$–intercept and $b$ is called its $y$–intercept.

(2) It can be proved that two lines with slopes $m_1$ and $m_2$ are parallel if $m_1 = m_2$, while they are perpendicular if $m_1 m_2 = -1$. 
2 Systems of Equations

Many problems in business and economics lead to what are called systems of linear equations. Consider for example, the following situation.

The owner of a television store wants to expand his business by buying and displaying two new types of television sets that have recently appeared on the market. Each television set of the first type costs $300 and each set of the second type costs $400. Each of the first type of set occupies 4 square feet of floor space, whereas each set of the second type occupies 5 square feet. If the owner has only $2000 available for this extension and 26 square feet of floor space, how many sets of each type should be bought and displayed to make full use of the available capital and space?

In this problem, we have two unknowns, namely, the number of the television sets of the first type, and that of the second type. Let $x$ denote the number of television sets of the first type the owner should buy, and $y$ the number of sets of the second type the owner should buy. Then in consideration of the cost, we should have

$$300x + 400y = 2000,$$

where $300x$ is the cost to buy $x$ sets of the first type of television sets, and $400y$ is the cost of buying $y$ sets of the second type. The total cost, by requirement, should be the money available, i.e. $300x + 400y = 2000$, that is exactly the equation we need to use.

Similarly, in consideration of the space, we should have

$$4x + 5y = 26$$

Thus the unknowns $x$ and $y$ are determined by the system of linear equations

$$
\begin{align*}
300x + 400y &= 2000 \\
4x + 5y &= 26.
\end{align*}
$$

To find $x$ and $y$, we need to solve this equation system. The method is to first eliminate one variable (or unknown), either $x$ or $y$, from the two equations; this allows determination of the value of the other variable.

Following this method, from the second equation in the above system, we have

$$4x = 26 - 5y$$

$$x = \frac{26 - 5y}{4}$$

We substitute this expression of $x$ into the first equation to obtain an equation involving $y$ only:
\[
300 \left( \frac{26 - 5y}{4} \right) + 400y = 2000 \\
75(26 - 5y) + 400y = 2000 \\
1950 - 375y + 400y = 2000 \\
25y = 2000 - 1950 = 50 \\
y = \frac{50}{25} = 2.
\]

Now we can find \(x\) by

\[
x = \frac{26 - 5y}{4} = \frac{26 - 5 \times 2}{4} = \frac{16}{4} = 4.
\]

Thus, to make full use of the available capital and space, the owner of the TV store should buy 4 sets of the first type of TV, and 2 sets of the second type.

In general, a \textbf{system of linear equations} of two variable \(x\) and \(y\) consists of two equations of the type

\[
a_1x + b_1y = c_1 \\
a_2x + b_2y = c_2
\]

where \(a_1, b_1, c_1, a_2, b_2\) and \(c_2\) are given constants. The \textbf{solution} of the system is the set of values of \(x\) and \(y\) that satisfy both equations.

Recall that, the linear equation \(a_1x + b_1y = c_1\) represents a straight line in the \(xy\)-plane, and \(a_2x + b_2y = c_2\) represents another straight line. A solution \(x\) and \(y\) of the system satisfies both equations, and therefore, the point \((x, y)\) lies on both lines, i.e., it is the intersection point of the straight lines. This geometric explanation of solutions to systems of two linear equations helps us to determine the number of such solutions even without actually solving the system. We know that for any two straight lines in the \(xy\)-plane, we have only three possibilities:

(a) They are not parallel, and hence intersect at exactly one point;

(b) They are parallel but are not the same, and hence they do not intersect.

(c) They coincide, and therefore intersect everywhere along the straight line.

In case (a), the system has exactly one solution, in case (b), there is no solution, while in case (c), there are infinitely many solutions.
Example 1  Solve the system
\[
\begin{align*}
  \frac{x-y}{3} &= \frac{y-1}{4} \\
  \frac{4x-5y}{7} &= x - 7
\end{align*}
\]

Solution  The system is not given in the simplest form. We first get rid of the fractions in the given equations. We multiply both sides of the first equation by 12, and obtain
\[
\begin{align*}
  4(x - y) &= 3(y - 1) \\
  4x - 4y &= 3y - 3 \\
  4x - 7y &= -3
\end{align*}
\]

Multiplying both sides of the second equation in the given system by 7 and simplifying, we obtain
\[
\begin{align*}
  4x - 5y &= 7(x - 7) = 7x - 49 \\
  -3x - 5y &= -49 \\
  3x + 5y &= 49
\end{align*}
\]

Thus the original system becomes
\[
\begin{align*}
  4x - 7y &= -3 \\
  3x + 5y &= 49
\end{align*}
\]

From the first equation of the simplified system, we obtain
\[
\begin{align*}
  4x &= 7y - 3 \\
  x &= \frac{7y - 3}{4}
\end{align*}
\]

Substituting this into the second equation of the simplified system, we obtain
\[
\begin{align*}
  3 \left( \frac{7y - 3}{4} \right) + 5y &= 49 \\
  3(7y - 3) + 20y &= 196 \\
  21y - 9 + 20y &= 196 \\
  41y &= 196 + 9 \\
  y &= \frac{205}{41} = 5
\end{align*}
\]

Hence,
\[
x = \frac{7y - 3}{4} = \frac{7 \times 5 - 3}{4} = \frac{35 - 3}{4} = \frac{32}{4} = 8
\]
Thus the solution is $x = 8, y = 5$. □

**Example 2.** Solve the system

\[
\begin{align*}
  x + 2y &= 4 \\
  3x + 6y &= 8
\end{align*}
\]

**Solution** From the first equation we get

\[x = 4 - 2y\]

Substituting this into the second equation we obtain

\[
\begin{align*}
  3(4 - 2y) + 6y - 8 &= 0 \\
  12 - 6y + 6y - 8 &= 0 \\
  4 &= 0
\end{align*}
\]

which is impossible. This means there is no solution to the system. □

**Remark.** It is easy to check that the two equations in Example 2 represents two parallel lines which do not intersect each other. Therefore there cannot be a solution to it.

**Example 3.** Solve the system

\[
\begin{align*}
  2x - 3y &= 6 \\
  \frac{x}{3} - \frac{y}{2} &= 1
\end{align*}
\]

**Solution** From the first equation one obtains

\[x = \frac{6 + 3y}{2}.
\]

Substitute this into the second equation,

\[
\begin{align*}
  \frac{1}{3} \left( \frac{6 + 3y}{2} \right) - \frac{y}{2} &= 1 \\
  \frac{6 + 3y}{6} - \frac{y}{2} &= 1 \\
  \frac{6}{6} + \frac{3y}{6} - \frac{y}{2} &= 1 \\
  1 &= 1
\end{align*}
\]

This means that the equation is true for any value of $y$, while the $x$ variable must be related to $y$ by

\[x = \frac{6 + 3y}{2},\]
i.e. \((x, y)\) must be on the line \(x = \frac{6 + 3y}{2}\), or \(2x - 3y = 6\). Therefore, we have infinitely many solutions to this system. The solutions can be expressed by

\[
x = \frac{6 + 3t}{2}, \quad y = t,
\]

where \(t\) is an arbitrary number.

Indeed, in Example 3, the two equations are equivalent (or they represent the same line). If we multiply the second equation by 6, we obtain

\[
\begin{align*}
6 \left( \frac{x}{3} \right) - 6 \left( \frac{y}{2} \right) &= 6 \\
2x - 3y &= 6
\end{align*}
\]

which is exactly the first equation.
3 Applications to Business Analysis

1. Linear Cost Model

In the production of any commodity by a firm, there are two types of costs involved; these are known as fixed costs and variable costs. Fixed costs are costs that have to be met no matter how much or how little of the commodity is produced; that is, they do not depend on the level of production. Examples of fixed costs are rents, interest on loans, and management salaries.

Variable costs are costs that depend on the level of production, that is, on the amount of commodity produced. Material costs and labour costs are examples of variable costs. The total cost is given by

\[ \text{Total Cost} = \text{Variable Costs} + \text{Fixed Costs} \]

In a linear cost model, one assumes that the variable costs per unit of commodity is constant, and let us denote it by \( m \) (dollars). Then the total variable costs of producing \( x \) units of commodity is \( mx \) (dollars). If the fixed costs are \( b \) dollars, then the total cost \( y_c \) (in dollars) of producing \( x \) units is given by

\[ y_c = mx + b \]

Example 1 The variable cost of processing 1 pound of coffee beans is 50 cents and the fixed costs per day are $300. Find the cost of processing 1000 pounds of coffee beans in one day.

Solution Total variable costs = 0.5 \times 1000. Fixed cost = 300. Therefore,

\[ \text{Total cost} = 0.5 \times 1000 + 300 = 800 \text{ (dollars)}. \]

Example 2. The cost of manufacturing 10 typewriters per day is $350, while it costs $600 to produce 20 typewriters per day. Assuming a linear cost model, determine the relationship representing the total cost \( y_c \) of producing \( x \) typewriters per day.

Solution Since it is a linear cost model,

\[ y_c = mx + b \]
where $b$ is the fixed cost, $m$ is the variable cost per commodity. By the conditions given, $y_c = 350$ when $x = 10$ and $y_c = 600$ when $x = 20$. We use these to determine $m$ and $b$.

\[
\begin{align*}
350 &= m \times 10 + b \\
600 &= m \times 20 + b
\end{align*}
\]

Subtracting the first equation from the second we obtain

\[
600 - 350 = (m \times 20 + b) - (m \times 10 + b)
\]

\[
250 = 10m, \quad m = \frac{250}{10} = 25
\]

Substituting this into the first equation we have

\[
\begin{align*}
350 &= 25 \times 10 + b = 250 + b \\
b &= 350 - 250 = 100.
\end{align*}
\]

Hence

\[
y_c = 25x + 100.
\]

2. Break-even Analysis

If the total cost $y_c$ of production exceeds the revenue $y_R$ obtained from the sales, then a business is running at a loss. On the other hand, if the revenue exceeds the costs there is a profit. If the cost of production equals the revenue obtained from the sales, there is no profit or loss, so the business breaks even. The number of units produced and sold in this case is called the break-even point.

Example 3  For a watch maker, the cost of labour and materials per watch is $15 and the fixed costs are $2000 per day. If each watch sells for $20, how many watches should be produced and sold each day to guarantee that the business breaks even?

Solution  The cost of producing $x$ watches per day is

\[
y_c = 15x + 2000
\]

The revenue is

\[
y_R = 20x.
\]

To break-even, we need $y_c = y_R$, i.e.

\[
15x + 2000 = 20x.
\]
\[
5x = \frac{2000}{5} = 400.
\]

Hence 400 watches should be produced and sold each day to break even. \qed

**Example 4** Suppose the total daily cost (in dollars) of producing \(x\) chairs is given by

\[
y_c = 2.5x + 300,
\]

and it is known that at least 150 chairs can be sold each day. What price should be charged to guarantee no loss?

**Solution** The total cost of producing 150 chairs per day is

\[
y_c = 2.5 \times 150 + 300 = 675
\]

If the price is \(p\) dollars per chair, then the revenue of selling 150 chairs is 150\(p\).

To break even, we should have

\[
150p = 675
\]

\[
p = \frac{675}{150} = 4.50.
\]

Thus, the price should be at least 4.50 dollars per chair to guarantee no loss. \qed

3. **Supply and Demand.**

The laws of demand and supply are two of the fundamental relationships in any economic analysis. The quantity \(x\) of any commodity that will be purchased by consumers depends on the price at which that commodity is made available; usually, if the price is decreased, then the demand will increase. A relationship that specifies the amount of a particular commodity that consumers are willing to buy at various price levels is called the **law of demand.** The simplest law is a linear relation of the type

\[
p = mx + b
\]

where \(p\) is the price per unit of the commodity, \(m\) and \(b\) are constants. The graph of the demand law is called the **demand curve.** In this linear case, it is just a straight line. \(m\) usually has a negative value to represent that demand increases when price decreases.
A relationship specifying the amount of any commodity that manufactures (or sellers) can make available in the market at various prices is called the supply law, its graph is called the supply curve. A linear supply law has the form

\[ p = m_1 x + b_1 \]

where \( p \) is the price per unit of the commodity, \( x \) is the number of units of the commodity and \( m_1, b_1 \) are constants, but this time, \( m_1 \) is usually positive (why?).

**Example 5.** A dealer can sell 20 electric shavers per day at $25 per shaver, but he can sell 30 shavers if he charges $20 per shaver. Determine the demand equation, assuming it is linear.

**Solution** The equation has the form

\[ p = mx + b \]

and we need to determine \( m \) and \( b \).

We know \( x = 20 \) when \( p = 25 \) and \( x = 30 \) when \( p = 20 \). i.e.

\[
\begin{align*}
25 &= m(20) + b \\
20 &= m(30) + b
\end{align*}
\]

Subtracting the first equation from the second, we obtain

\[ 10m = -5 \quad m = \frac{-5}{10} = -0.5 \]

Substituting this into the first equation, we have

\[
\begin{align*}
25 &= -0.5 \times 20 + b = -10 + b \\
b &= 35
\end{align*}
\]

Thus the demand law is

\[ p = -0.5x + 35. \]

4. Market Equilibrium

If the price of a certain commodity is too high, consumers will not purchase it, whereas if the price is too low, suppliers will not sell it. In a competitive market, when the price per unit depends only on the quantity demanded and the supply available, there is always a tendency for the price to adjust itself so that the quantity demanded by the purchasers matches the quantity which suppliers are willing to supply. **Market equilibrium** is said to occur at the price when the quantity demanded is equal to the quantity supplied.
Example 6  Determine the equilibrium price and quantity for the following demand and supply laws.

\[ D : \quad p = 25 - 2x \]
\[ S : \quad p = 3x + 5 \]

Solution.  At equilibrium, the price \( p \) and quantity \( x \) in both equations are the same, i.e. they satisfy both equations. Hence \( p \) and \( x \) are solutions of the system of linear equations. Subtract the demand equation from the supply equation, it results

\[ 3x + 5 - (25 - 2x) = p - p = 0 \]
\[ 5x - 20 = 0, \quad 5x = 20, \quad x = \frac{20}{5} = 4. \]

Substitute this back to the demand equation,

\[ p = 25 - 2 \times 4 = 25 - 8 = 17. \]

Hence the equilibrium quantity and price are \( x = 4 \) and \( p = 17 \), respectively. \( \square \)
4 Functions

Definition Let $X$ and $Y$ be two nonempty sets. Then a function from $X$ to $Y$ is a rule that assigns to each element $x$ in $X$ a unique $y$ in $Y$. If a function assigns $y$ to a particular $x$, we say $y$ if the value of the function at $x$.

A function is generally denoted by a single letter such as $f, g, F,$ or $G$. If $f$ denotes a given function, the set $X$ for whose elements unique values in $Y$ are assigned is called the domain of the function $f$, and is often denoted by $D_f$. The corresponding set of values in $Y$ is called the range of the function $f$ and is often denoted by $R_f$.

Example 1 Let $X$ be the set of students in a class. Let $f$ be the rule which assigns to each student his or her age. Then $f$ is a function, the domain is the set of all students in the class and the range is the set of all the ages. (For example, $R_f$ might be the set $\{18, 19, 20, 22, 40\}$).

If a function $f$ assigns a value $y$ to a certain $x$ in the domain, we write $y = f(x)$. We read $f(x)$ as “$f$ of $x$”; it is called the value of $f$ at $x$. Note that $f(x)$ is NOT the product of $f$ and $x$.

If a function $f$ is expressed by a relation of the type $y = f(x)$, then $x$ is called the independent variable or argument of $f$, and $y$ is called the dependent variable.

In most cases, the independent and dependent variables are numbers and the function is expressed by an algebraic formula. For example,

\[
\begin{align*}
f(x) &= 5x^2 - 7x + 2 \\
g(p) &= 2p^3 + \frac{7}{p + 1}
\end{align*}
\]

Example 2 Given $f(x) = 2x^2 - 5x + 1$, find the value of $f$ when $x = 3, x = -2, x = a$; i.e. find $f(3), f(-2)$ and $f(a)$.

Solution

\[
\begin{align*}
f(3) &= 2(3)^2 - 5(3) + 1 = 18 - 15 + 1 = 4 \\
f(-2) &= 2(-2)^2 - 5(-2) + 1 = 8 + 10 + 1 = 19 \\
f(a) &= 5a^2 - 5a + 1
\end{align*}
\]

When a function is given by an algebraic formula, as in Example 2, its domain is
often not stated explicitly. In such cases, it is understood that the domain consists of all the values of the argument for which the given rule makes sense.

In general, when finding the domain of a function, we must bear these two conditions in mind:

(a) Any expression underneath a square root cannot be negative.
(b) The denominator of any fraction cannot be zero.

**Example 3** Find the domain of $g$, where

$$g(x) = \frac{\sqrt{x-1}}{x-2}$$

**Solution.** For the fraction to make sense, we require $x \neq 2$. For the square root to make sense, we need $x - 1 \geq 0$, i.e. $x \geq 1$.

Since we must meet both requirements to make the entire expression well defined, we must have $x \geq 1$ and $x \neq 2$. That is, the domain consists of all the numbers not less than 1 except the number 2. This set can be denoted by

$$D_g = \{x | x \geq 1, x \neq 2\}.$$

When the domain and range of a function $y = f(x)$ are both sets of real numbers, the function can be represented by its **graph**, which is obtained by plotting all of the points $(x, y)$, where $x$ belongs to the domain of $f$ and $y = f(x)$, treating $x$ and $y$ as Cartesian coordinates.

**Example 4** For the function $f(x) = 2 + 0.5x^2$, the domain is the set of all real numbers. Some of the values of this function are shown in the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = f(x)$</td>
<td>2</td>
<td>2.5</td>
<td>4</td>
<td>6.5</td>
<td>10</td>
<td>2.5</td>
<td>4</td>
<td>6.5</td>
<td>10</td>
</tr>
</tbody>
</table>

The points corresponding to the values of $x$ and $y$ in the table are plotted as dots in the following diagram. The graph of this function is shown as the $U$–shaped curve passing through the dots.
In applied problems, it is often important to construct an algebraic function from certain verbal information so that mathematical analysis can be used.

**Example 5** (Cost of Telephone Link). A telephone link is to be constructed between two towns that are situated on opposite banks of a river at points A and B. The width of the river is 1 kilometer and B lies 2 kilometers downstream from A. It costs $C$ dollars per kilometer to construct a line over land and $20$ dollars per kilometer under water. The telephone line will follow the river bank from A for a distance of $x$ kilometers and then will cross the river diagonally in a straight line directly to $B$. Determine the total cost of the line as a function of $x$.

**Solution.** As indicated in the diagram, the telephone line will have $x$ kilometers over land which is the line segment from A to C. The line segment from C to B is the part of the telephone line under water.

By the Pythagorean theorem to the triangle $BCD$,

$$BC^2 = BC'^2 + CD^2.$$

But we know $BD = 1, CD = (2 - x)$. Therefore

$$BC^2 = 1^2 + (2 - x)^2 = 5 - 4x + x^2$$

$$BC = \sqrt{5 - 4x + x^2}$$

Therefore we have $x$ kilometers of the telephone line overland, and $\sqrt{5 - 4x + x^2}$ kilometers of the telephone line under water. The total cost is

$$y = cx + 2c\sqrt{5 - 4x + x^2}.$$
Example 6 (Electricity Cost Function) Electricity is charged to consumers at the rate of 10 cents per unit for the first 50 units and 3 cents per unit for amount in excess of this. Find the function $c(x)$ that gives the cost of using $x$ units of electricity.

Solution For $x \leq 50$, each unit costs 10 cents, so the total cost of $x$ unit is $10x$ cents. Therefore, 

$$c(x) = 10x \text{ for } x \leq 50.$$ 

When $x > 50$, the first 50 unit costs $10 \times 50 = 500$ (cents); the number of the excess units is $x - 50$, which costs $3(x - 50)$, as the rate is 3 cents per unit for these units. Thus the total cost when $x > 50$ is 

$$c(x) = 500 + 3(x - 50) = 350 + 3x$$

We can write $c(x)$ in the form 

$$c(x) = \begin{cases} 10x & \text{if } x \leq 50 \\ 350 + 3x & \text{if } x > 50. \end{cases}$$

The graph of $y = c(x)$ is shown in the following diagram. Observe how the nature of the graph changes at $x = 50$, where one formula takes over from the other.

A function $f$ defined by the relation 

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad (a_n \neq 0)$$

where $a_0, a_1, \cdots, a_n$ are constants and $n$ is a nonnegative integer, is said to be a polynomial function of degree $n$. For example, 

$$f(x) = 3x^7 - 5x^4 + 2x - 1 \quad \text{and} \quad g(x) = x^3 + 7x^2 - 5x + 3$$

are polynomial functions of degree 7 and 3, respectively.

If the degree of a polynomial function is 1, then it is a linear function which can be expressed in the form 

$$f(x) = mx + b \quad (m \neq 0).$$
We know the graph of \( y = mx + b \) is a straight line, \( m \) is the slope and \( b \) is the \( y \)-intercept of the line.

If the degree of the polynomial function is 2, then the function is called a **quadratic function**. The general quadratic function may be expressed as

\[
g(x) = ax^2 + bx + c \quad (a \neq 0)
\]

where \( a, b \) and \( c \) are constants.

If a function can be expressed as the quotient of two polynomial functions, then it is called a **rational function**. Examples of rational functions are

\[
f(x) = \frac{x^2 - 9}{x - 4}, \quad g(x) = \frac{x^3 + x^2 - 1}{x^6 + 1}.
\]

If the value \( f(x) \) of a function \( f \) is found by a finite number of algebraic operations, \( f \) is called an **algebraic function**. The algebraic operations are addition, subtraction, multiplication, division, raising to powers, and extracting roots. For example,

\[
f(x) = \frac{(2x + 1)^2 - \sqrt{x^4 + 5}}{(x^2 + 1)^2}, \quad g(x) = (2x - 1)^{-\frac{1}{5}} + 5x^{\frac{3}{5}}
\]

are algebraic functions.

Apart form algebraic functions, there are other functions called **transcendental functions**. Examples of transcendental functions are logarithmic functions, and exponential functions, which will be discussed later.
5 Quadratic Functions and Parabolas

A function of the form

\[ f(x) = ax^2 + bx + c \quad (a \neq 0) \]

where \( a, b, \) and \( c \) are constants, is called a **quadratic function**. The domain of \( f \) is the set of all real numbers.

The simplest quadratic function is obtained by setting \( b \) and \( c \) equal to zero, in which case we obtain \( f(x) = ax^2 \). Typical graphs of this function are shown below.

The lowest point on the graph when \( a > 0 \) occurs at the origin, while the origin is the highest point when \( a < 0 \). Each of these graphs is called a **parabola**. The origin (which is the lowest or highest point) is called the **vertex** of the parabola.

When \( b \) and \( c \) are not necessarily zero, we can rewrite

\[
\begin{align*}
  f(x) &= ax^2 + bx + c \quad \text{in the form} \\
  f(x) &= ax^2 + bx + c = a \left( x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}
\end{align*}
\]

The graph of \( f(x) \) has the equation \( y = ax^2 + bx + c \) which, by the above identity, is equivalent to,

\[
y - \left( c - \frac{b^2}{4a} \right) = a \left( x + \frac{b}{2a} \right)^2.
\]

This implies that the graph of \( f(x) = ax^2 + bx + c \) is identical in shape and size to the graph of \( y = ax^2 \); the only difference is that the vertex of \( f(x) = ax^2 + bx + c \) is shifted to the point with coordinates \( \left( \frac{-b}{2a}, c - \frac{b^2}{4a} \right) \). Note how these coordinates are related to the new equivalent equation given above.
As indicated above, the vertex of a parabola represents the lowest point when 
\( a > 0 \) or the highest point when \( a < 0 \). It follows, therefore, that for \( a > 0 \), the function \( f(x) = ax^2 + bx + c \) takes its minimum value at the vertex, i.e., \( f(x) \) is smallest when \( x = -\frac{b}{2a} \), and the smallest value is \( f\left(-\frac{b}{2a}\right) = c - \frac{b^2}{4a} \).

Correspondingly, when \( a < 0 \), the function \( f(x) = ax^2 + bx + c \) takes its largest value when \( x = -\frac{b}{2a} \), and the largest value is \( f\left(-\frac{b}{2a}\right) = c - \frac{b^2}{4a} \).

Finding maximum and minimum values of certain functions arises very frequently in applications. Our above discussion is useful when we want to find the maximum and minimum values of quadratic functions.

Let us look at some examples that quadratic functions are used in practical problems.

**Example 1 (Fencing)** A farmer has 200 yards of fencing material with which to enclose a rectangular field. One side of the field can make use of a fence that already exists. What is the maximum area that can be enclosed?

**Solution** Let the sides of the field to be fenced be denoted by \( x \) and \( y \), as indicated by the diagram.

Then \( 2x + y = 200 \) and the area is \( A = xy \).

From \( 2x + y = 200 \) we deduce \( y = 200 - 2x \). Hence

\[
A = x(200 - 2x) = -2x^2 + 200x
\]

Thus the area \( A \) is a quadratic function of \( x \), with \( a = 2 < 0, b = 200, c = 0 \). We know the maximum of the function (the largest value of the function) occurs at \( x = -\frac{b}{2a} = -\frac{200}{2(-2)} = \frac{200}{4} = 50 \), and the maximum is

\[
A(50) = -2(50)^2 + 200(50) = -5000 + 10000 = 5000
\]
Thus the maximum area that can be enclosed is 5000 square yards (when the dimensions are $x = 50$, $y = 200 - 2x = 100$).

Example 2 (Pricing Decision). The demand per month, $x$, for a certain commodity at a price $p$ dollars per unit is given by the relation

$$x = 1350 - 45p.$$ 

The cost of labor and material to manufacture this commodity is $5$ per unit and the fixed costs are $2000$ per month. What price $p$ per unit should be charged to the consumers to obtain a maximum monthly profit?

**Solution** The total cost $C$ (in dollars) of producing $x$ units per month is

$$C = \text{Variable costs} + \text{Fixed costs} = 5x + 2000$$

The number of units produced per month, $x$, should equal the demand per month, i.e.

$$x = 1350 - 45p$$

Therefore

$$C = 5x + 2000 = 5(1350 - 45p) + 2000 = 8750 - 225p$$

The revenue $R$ obtained by selling $x$ units at $p$ dollars per unit is

$$R = px = p(1350 - 45p) = 1350p - 45p^2$$

Thus, the profit $P$ can be expressed as

$$P = R - C = (1350p - 45p^2) - (8750 - 225p)$$

$$= -45p^2 + 1575p - 8750$$

which is a quadratic function in $p$, with $a = -45 < 0$, $b = 1575$, $c = -8750$. We know the maximum of this function occurs at $p = -\frac{b}{2a} = -\frac{1575}{2(-45)} = \frac{1575}{90} = 17.5$.

Thus the price of $p = $17.5 per unit should be charged to obtain a maximum profit. 

Note that the maximum profit in Example 2 is $P(17.5) = 5031.25$. 

\[ \square \]
6 More Simple Functions

Power Functions.

A function of the form
\[ f(x) = ax^n \]
where \( a \) and \( n \) are nonzero constants, is called a \textbf{power function}.

Let us first look at a few specific power functions.

1. \( n = 1, f(x) = ax \). This is a linear function, its graph is a straight line, passing through the origin, with slope \( a \).

2. \( n = 2, f(x) = ax^2 \). This is a quadratic function, its graph is a parabola with vertex at the origin.

3. \( n = \frac{1}{2}, f(x) = ax^{\frac{1}{2}} = a\sqrt{x} \). The domain of the function is the set of nonnegative real numbers. Its graph has the equation \( y = a\sqrt{x} \). Squaring both sides of the equation we obtain \( y^2 = a^2x \), i.e. \( x = \frac{1}{a^2}y^2 = a'y^2 \), where \( a' = \frac{1}{a^2} > 0 \) is a constant. Comparing \( x = a'y^2 \) with \( y = a'x^2 \), we see we can obtain one equation from the other by interchanging the positions of \( x \) and \( y \) in the equations. As the graph of \( y = a'x^2 \) is a parabola as indicated in the following diagram (note \( a' = \frac{1}{a^2} > 0 \)),

it is not hard to realize that the graph of \( x = a'y^2 \) is also a parabola which is the same in shape and in size as the one given by \( y = a'x^2 \), but is symmetric about the \( x \)-axis, instead of the \( y \)-axis.
As \( x = a'y^2 \) is equivalent to \( y^2 = a'^2x \), and the later is obtained from \( y = a√x \), the graph of \( y = a√x \) should be the same as that of \( x = a'y^2 \) except that in the equation \( x = a'y^2 \), there is no restriction for the values that \( y \) can take, but in the equation \( y = a√x \), \( y \) takes only nonnegative values if \( a > 0 \) and it takes only nonpositive values if \( a < 0 \). Thus, the graph of \( y = a√x \) is the upper half parabola given by \( x = a'y^2 \) if \( a > 0 \), and it is the lower half parabola given by \( x = a'y^2 \) if \( a > 0 \).

4. \( n = -1, f(x) = ax^{-1} = \frac{a}{x} \). The domain of \( f \) is the set of all real numbers except zero. The graph of this function consists of two curves, and is again affected by whether \( a > 0 \) or \( a < 0 \).

The curves approach the \( x \)-axis as \( x \) becomes large positive or large negative, and we say the \( x \)-axis is a **horizontal asymptote** to the graph. As \( x \) becomes small positive or small negative, the curves approach the \( y \)-axis, and we say the \( y \)-axis is a **vertical asymptote** to the graph.
5. \( n = 3, f(x) = ax^3 \). The domain of \( f \) is the set of all real numbers. The graph consists of one curve, called a \textbf{cubic curve}, as shown in the following diagram.

The following diagram provides a comparison of the graphs of \( y = ax^n \) for various values of \( n \) with \( a > 0 \), where the graphs are drawn only for the quadrant in which \( x \) and \( y \) are nonnegative (as occurs mostly in business and economic applications).

We see that all the graphs pass through the point \((1, a)\). When \( n > 1 \), the graph rises as we move to the right and, moreover, rises more and more steeply as \( x \) increases. The case \( n = 1 \) corresponds to a straight line. When \( 0 < n < 1 \), the graph still rises as we move to the right, but it rises less steeply as \( x \) increases. When \( n < 0 \), the graph falls as we move to the right and is asymptotic to the \( x \)-axis and \( y \)-axis.

\textbf{Example 1.} A firm wants to obtain a total revenue of $500 per day regardless of the price it charges for its product. Find the demand relation and graph the demand curve.

\textbf{Solution} Let \( p \) denote the price per unit and \( x \) the number of units the firm can sell per day. Then the revenue

\[ y_R = px. \]

To have \( y_R = 500 \) always, we get

\[ 500 = px, \quad \text{or} \]
\[ p = \frac{500}{x}. \]

To obtain the graph, we use the following table

\[
\begin{array}{c|cccccc}
 x & 25 & 50 & 100 & 125 & 250 & 500 \\
p & 20 & 10 & 5 & 4 & 2 & 1 \\
\end{array}
\]

Note that we have restricted the graph to the first quadrant because the price \( p \) and quantity sold \( x \) can only take nonnegative numbers.
7 Combinations of Functions

A variety of situations arise in which we have to combine two or more functions in one of several ways to get new functions. For example, let \( f(t) \) and \( g(t) \) denote the income of a person from two different sources at time \( t \); then the combined income from the two sources is \( f(t) + g(t) \), which is a new function formed as the sum of \( f \) and \( g \). If \( C(x) \) denotes the cost of producing \( x \) units of a certain commodity and \( R(x) \) is the revenue obtained from the sale of \( x \) units, then we know the profit \( P(x) \) obtained by producing and selling \( x \) units is given by \( P(x) = R(x) - C(x) \), which is the difference of the functions \( R \) and \( C \). In other situations, the product or quotient of two functions also arise. These lead to the following abstract definitions.

**Definition** Given two functions \( f \) and \( g \), the sum, difference, product, and quotient of \( f \) and \( g \) are defined as follows:

- **Sum**: \((f + g)(x) = f(x) + g(x)\)
- **Difference**: \((f - g)(x) = f(x) - g(x)\)
- **Product**: \((f \cdot g)(x) = g(x) \cdot g(x)\)
- **Quotient**: \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\), provided \( g(x) \neq 0 \)

The domains of the sum, difference, and product functions are all equal to the common part of the domains of \( f \) and \( g \). In the case of the quotient function, the domain is the common part of the domains of \( f \) and \( g \) except for those values of \( x \) for which \( g(x) = 0 \).

Note that apart from making new, and usually more complicated functions form simpler ones, the combination of functions can also be used to break a given complicated function into simpler ones. For example, we can regard \( f(x) = x^3 + x^2 \) as the sum of the two simpler functions \( g(x) = x^3 \) and \( h(x) = x^2 \).

**Example 1.** Let \( f(x) = \frac{1}{x-1} \) and \( g(x) = \sqrt{x} \). Find \( f + g \), \( f \cdot g \) and \( g/f \). Determine the domain in each case.

**Solution** \((f + g)(x) = f(x) + g(x) = \frac{1}{x-1} + \sqrt{x}\).

The domain consists of all the values of \( x \) satisfying both \( x \neq 1 \) and \( x \geq 0 \). That is the set of all nonnegative numbers except \( x = 1 \).

\((f \cdot g)(x) = f(x) \cdot g(x) = \left(\frac{1}{x-1}\right) \cdot \sqrt{x} = \frac{\sqrt{x}}{x-1}\). The domain is the same as that of \( f + g \) given above.
\[
\left( \frac{g}{f} \right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{x}}{\frac{1}{x-1}} = \frac{\sqrt{x} \cdot (x-1)}{(\frac{1}{x-1}) (x-1)} = \sqrt{x} (x-1)
\]

The final expression makes sense whenever \(x\) is nonnegative. In spite of this, it is still necessary to exclude \(x = 1\) from the domain, as \(g/f\) is defined only at points where both \(f\) and \(g\) are defined. Thus the domain is the same as \(f + g\). \(\Box\)

Another way in which two functions can be combined to yield a third function is called the composition of functions. Consider the following situation.

The monthly revenue \(R\) of a firm depends on the number \(x\) of the units that is produced and sold. In general, we can say \(R = f(x)\). Usually, the number \(x\) of units it can sell depends on the price \(p\) per unit it charges the consumers, so that \(x = g(p)\). If we substitute \(x = g(p)\) into the relation \(R = f(x)\), we obtain

\[
R = f(x) = f(g(p)).
\]

This gives \(R\) as a function of the price \(p\). This way of forming the new function of \(p\), \(f(g(p))\), is called the composition of functions. A formal definition is given below.

**Definition** Let \(f\) and \(g\) be two functions. Then the composition function \(f \circ g\) (read \(f\) circle \(g\)) is defined by

\[
(f \circ g)(x) = f(g(x)).
\]

The domain of \(f \circ g\) is a part of the domain of \(g\) which consists of all the values of \(x\) that makes \(g(x)\) a value in the domain of \(f\), i.e. \(D_{f \circ g} = \{x | x \in D_g \text{ and } g(x) \in D_f\}\).

**Example 2.** Let \(f(x) = \frac{1}{x+2}\) and \(g(x) = \sqrt{x}\). Find \((f \circ g)(x), (g \circ f)(x)\) and determine the domains.

**Solution** \((f \circ g)(x) = f(g(x)) = \frac{1}{g(x)+2} = \frac{1}{\sqrt{x}+2}\).

The domain consists of all nonnegative real numbers.

\[
(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{\frac{1}{x+2}} = \frac{1}{\sqrt{x+2}}
\]

The domain consists of all \(x\) satisfying \(x + 2 > 0\), i.e. \(x > -2\), or \(D_{g \circ f} = \{x | x > -2\}\). \(\Box\)
Example 3. The monthly revenue \( R \) obtained by selling deluxe model shoes is a function of the demand \( x \) in the market. It is observed that, as a function of price \( p \) per pair, the monthly revenue and demand are

\[
R = 300p - 2p^2, \quad x = 300 - 2p.
\]

How does \( R \) depend on \( x \)?

Solution From \( x = 300 - 2p \) we obtain \( 2p = 300 - x \), \( p = \frac{300-x}{2} = 150 - \frac{1}{2}x \).

Substituting this expression of \( p \) (as a function of \( x \)) into the expression of \( R \) (as a function of \( p \))

\[
R = 300p - 2p^2
\]

we obtain (a composite function)

\[
R = 300(150 - \frac{1}{2}x) - 2(150 - \frac{1}{2}x)^2 \\
= 300 \times 150 - 300 \left( \frac{1}{2}x \right) - 2 \left[ 150^2 - 2 \times 150 \times \frac{1}{2}x + \left( \frac{1}{2}x \right)^2 \right] \\
= -\frac{1}{2}x^2 + 150x
\]

That is, the relationship between \( R \) and \( x \) is

\[
R = -\frac{1}{2}x^2 + 150x.
\]

Example 4. Find \( f \) and \( g \) such that the function

\[
F(x) = \frac{1}{\sqrt{x^2 + 4}}
\]

is the composite function \( f \circ g \)

Solution If we let \( g(x) = \sqrt{x^2 + 4} \), then \( F(x) = \frac{1}{g(x)} \). Hence \( F(x) = (f \circ g)(x) \) where \( f(x) = \frac{1}{x} \).

But the solution is not unique. Some other choices of \( f \) and \( g \) are: \( g(x) = \frac{1}{x^2 + 4}, F(x) = \sqrt{g(x)}, f(x) = \sqrt{x}; g(x) = x^4 + 4, F(x) = \frac{1}{\sqrt{g(x)}}, f(x) = \frac{1}{\sqrt{x}} \).
8 Implicit Relations and Inverse Functions

When $y$ is a given function of $x$, that is, $y = f(x)$, then we often say that $y$ is an explicit function of the independent variable $x$. Examples of explicit functions are $y = 3x^2 + 5$, and $y = \frac{6x}{x^2 + 1}$.

Sometimes the fact that $y$ is a function of $x$ is expressed indirectly by means of some equation of the type $F(x, y) = 0$, in which both $x$ and $y$ appear as arguments of the function $F$. An equation of this type is called an implicit relation between $x$ and $y$.

**Example 1.** The equation $xy + 3y - 7 = 0$ gives an implicit relation between $x$ and $y$. Solving for $y$ we obtain

\[
(x + 3)y + 7 = 0 \\
(x + 3)y = 7 \\
y = \frac{7}{x + 3}
\]

This gives $y$ as an explicit function of $x$.

We can also solve for $x$.

\[
xy = 7 - 3y, \\
x = \frac{7 - 3y}{y} = \frac{7}{y} - 3
\]

Hence the implicit relation also gives $x$ as an explicit function of $y$.

**Example 2** Consider the implicit relation $x^2 + y^2 = 4$. In this case, we can again solve for $y$.

\[
y^2 = 4 - x^2 \\
y = \sqrt{4 - x^2} \text{ or } y = -\sqrt{4 - x^2}
\]

Thus the implicit relation $x^2 + y^2 = 4$ leads to two explicit functions,

\[
y = \sqrt{4 - x^2} \text{ and } y = -\sqrt{4 - x^2}
\]

**Example 3.** Consider the implicit relation $x^2 + \sqrt{1 + y^2} = 0$. We know $x^2 \geq 0$ always, and $\sqrt{1 + y^2} \geq 1$. Hence $x^2 + \sqrt{1 + y^2}$ is never less than 1. This shows that one can never find any value of $x$ and $y$ such that $x^2 + \sqrt{1 + y^2} = 0$. This means the implicit relation has no solution.
When the fact that $y$ is a function of $x$ is implied by some relation of the form $F(x, y) = 0$, we speak of $y$ as an *implicit function* of $x$. When the expression of $F(x, y)$ is complicated, it is usually not possible to solve from $F(x, y) = 0$ for $y$ as an explicit function of $x$, but sometimes one can use advanced mathematics to know certain relations do determine $y$ as functions of $x$. This, however, is beyond the scope of this unit.

Going back to Example 1, where we see that the relation $xy + 3y - 7 = 0$ gives $y$ as a function of $x$:

$$y = \frac{7}{x + 3}$$

It also gives $x$ as a function of $y$:

$$x = \frac{7}{y} - 3.$$

If we denote $f(x) = \frac{7}{x + 3}$, it is not difficult to see that the function $x = \frac{7}{y} - 3$ can be obtained by solving $y = f(x)$ for $x$.

$$y = \frac{7}{x + 3}, \ y(x + 3) = 7, \ x + 3 = \frac{7}{y}, \ x = \frac{7}{y} - 3.$$

In general, if one can solve from $y = f(x)$ for $x$ to obtain a function $x = g(y)$, then the function $g(y)$ is called the *inverse function* of $f$, and it is often denoted by $f^{-1}$, i.e. $f^{-1}(y) = g(y)$.

**Example 4.** Find the inverse of the function $f(x) = 3x + 1$.

**Solution** Let $y = 3x + 1$. Then $3x = y - 1, x = \frac{y - 1}{3}$.

Thus the inverse function is $f^{-1}(y) = \frac{y - 1}{3}$.

Note that not every function has an inverse. Consider, for example, the function $y = x^2$. Solving for $x$ in terms of $y$, we obtain

$$x = \pm \sqrt{y}.$$ 

Therefore, corresponding to each positive value of $y$, we have two corresponding values of $x : x = \sqrt{y}$ and $x = -\sqrt{y}$. This shows that $x$ is not a function of $y$ (recall from the definition of a function $g$ from a set $Y$ to a set $X$: for each $y$ in $Y$, there is assigned a unique value $x = g(y)$.)

**Example 5** Solve the following implicit relation to express $x$ explicitly in terms of $y$. 

$$(y^2 + 1)x^2 + 6x + y^3 = 0$$
Solution  Regarding $y$ as a constant, we can view the given equation as a quadratic equation for $x$ of the form

$$ax^2 + bx + c = 0,$$

where $a = y^2 + 1, b = 6, c = y^3$.

We know the solution of $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$ 

Therefore, substituting $a = y^2 + 1, b = 6, c = y^3$ into these formulas, we have

$$x = \frac{-6 + \sqrt{36 - 4(y^2 + 1)y^3}}{2(y^2 + 1)} \quad \text{and} \quad x = \frac{-6 - \sqrt{36 - 4(y^2 + 1)y^3}}{2(y^2 + 1)}.$$ 

$\square$
9 Compound Interest

Consider a sum of money, say $100, that is invested at a fixed rate of interest, such as 6% per annum. After 1 year, the investment will have increased in value by 6% to $106. If the interest is compounded, then during the second year, this whole sum of $106 earns interest at 6%. Thus the value of the investment at the end of the second year will consist of the $106 existing at the beginning of that year plus 6% of $106 in interest, giving a total value of

\[ \$106 + (\$106)(0.06) = (\$106)(1 + 0.06) = (\$106)(1.06) = \$100 \times (1.06) \]

During the third year, the value increases by an amount of interest equal to 6% of $112.36, giving a total value at the end of that year equal to

\[ \$112.36 + \$112.36 \times 0.06 = \$112.36(1 + 0.06) = \$112.36(1.06) \]

\[ = \$100 \times (1.06)^2 \]

In general, the investment increases by a factor of 1.06 with each year that passes, so after \( n \) years, its value is $100 \times (1.06)^n$.

The above way of calculating the value of an investment growing at compound interest can be generalized to the general case, that is, if a sum \( P \) is invested at a rate of interest of \( R \) percent per annum, then the value of the investment after \( n \) years is given by the formula

\[ \text{Value after } n \text{ years} = P(1 + i)^n, \quad i = \frac{R}{100} \]

**Example 1 (Investment)** A sum of $200 is invested at 5% interest compounded annually. Find the value of the investment after 10 years.

**Solution** We have \( P = 200, \ R = 5, \ i = \frac{5}{100} = 0.05, \) and \( n = 10 \). Hence,

\[ \text{Value after 10 years} = P(1 + i)^n = 200(1 + 0.05)^{10} = 200 \times (1.05)^{10} = 200 \times (1.628895) = 325.78 \text{ (dollars)} \]

In some cases, interest is compounded more than once per year, for example semiannually (2 times per year), quarterly (4 times per year) or monthly (12 times
per year). In these cases, annual rate of interest $R$ percent which is usually quoted is called the nominal rate. If compounding occurs $k$ times per year with a nominal rate of interest $R$ percent, then the interest rate at each compounding is equal to $\frac{R}{k}$ percent, and if the number of compounding period is $n$, the compound interest formula becomes

$$\text{Value after } n \text{ periods} = P \left(1 + \frac{R}{100k}\right)^n$$

Thus, after $N$ years, we have $kN$ compounding periods, and

$$\text{Value after } N \text{ years} = P \left(1 + \frac{R}{100k}\right)^{kN}$$

**Example 2 (Monthly Compounding)** A sum of $2000 is invested at a nominal rate of interest of 9% compounded monthly. Find the value of the investment after 3 years.

**Solution** We have $k = 12$, $R = 9$ and $P = 2000$. After 3 years, there are $n = 3 \times k = 3 \times 12 = 36$ compounding periods. Hence

$$\text{Value after 3 years} = P \left(1 + \frac{R}{100k}\right)^n = 2000 \left(1 + \frac{9}{100 \times 12}\right)^{36} = 2000(1.0075)^{36} = 2617.29 \text{ (dollars)}.$$  

□

The effective rate of interest of an investment is defined as the annual rate that would provide the same growth if compounded once per year. Consider an investment that is compounded $k$ times per year at nominal rate of interest of $R\%$. Then the value of investment after 1 year is

$$P \left(1 + \frac{R}{100k}\right)^k.$$  

i.e. the investment grows by a factor $\left(1 + \frac{R}{100k}\right)^k$ in one year. If we use $i_{eff}$ to denote the effective rate of interest, the investment grows by a factor $(1 + i_{eff})$ each year. Thus we must have

$$1 + i_{eff} = \left(1 + \frac{R}{100k}\right)^k, \quad i_{eff} = \left(1 + \frac{R}{100k}\right)^k - 1$$

**Example 3** Which is better for the investor, 12% compounded monthly or 12.2% compounded quarterly?
Solution  We compute the effective rate for each of the two investment plan, the one with a larger effective rate is better for the investor.

For the first plan, $R = 12, k = 12$,

$$i_{eff} = \left(1 + \frac{12}{100 \times 12}\right)^{12} - 1 = (1.01)^{12} - 1 = 0.126825$$

For the second plan, $R = 12.2, k = 4$, so

$$i_{eff} = \left(1 + \frac{12.2}{100 \times 4}\right)^{4} - 1 = (1.0305)^{4} - 1 = 0.127696.$$ 

Therefore, the second plan, 12.2% compounded quarterly, is better for the investor. □
Exponential functions arise naturally in practical problems. Therefore, before we formally introduce exponential functions in the mathematical way, let us look at a few practical situations that exponential functions arise.

1. Continuous Compounding

Suppose a sum of $100 is invested at a nominal rate of interest of 8% compounded $k$ times per year. Then, after 4 years, the value of the investment is, according to the formula in the previous lecture,

\[
\text{Value after 4 years} = 100 \left(1 + \frac{8}{100k}\right)^{4k}
\]

\[
= 100 \left(1 + \frac{0.08}{k}\right)^{4k}
\]

We are interested in knowing how the value changes with $k$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>4</th>
<th>12</th>
<th>52</th>
<th>365</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value after 4 years</td>
<td>$136.05$</td>
<td>$137.28$</td>
<td>$137.57$</td>
<td>$137.68$</td>
<td>$137.71$</td>
<td>$137.71$</td>
</tr>
</tbody>
</table>

The above table shows that the value increases with $k$ but it does not seem to increase without bound, but rather approaches closer and closer to a certain value. To the nearest cent, there is no difference between compounding 365 times a year and 1000 times a year.

Because of this, we can envisage the possibility of what is called **continuous compounding**, namely, $k$ is allowed to become arbitrarily large; we say $k$ is allowed to approach infinity and we write this as $k \to \infty$. This corresponds to compounding the interest infinitely often during the year. With our $100 invested at the nominal rate of 8% per annum, continuous compounding gives a value of $137.71 after 4 years, the same value as daily compounding.

We still want to find a formula for the general case of continuous compounding. Let us denote $p = k/0.08$. Then the value of the investment in the above case, with $k$ times compounding per year, is

\[
100 \left(1 + \frac{0.08}{k}\right)^{4k} = 100 \left(1 + \frac{1}{p}\right)^{0.32p} = 100 \left[\left(1 + \frac{1}{p}\right)^p\right]^{0.32}
\]

The reason for writing it in this form is that as $k \to \infty$, then $p = \frac{k}{0.08}$ also
approached infinity, and the value of \((1 + \frac{1}{p})^p\) gets closer and closer to a certain important constant as \(p \to \infty\).

The following table gives the values of \((1 + \frac{1}{p})^p\) for a series of increasing large values of \(p\).

<table>
<thead>
<tr>
<th>(p)</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1 + \frac{1}{p})^p)</td>
<td>2.25</td>
<td>2.594</td>
<td>2.705</td>
<td>2.717</td>
<td>2.718</td>
<td></td>
</tr>
</tbody>
</table>

The eventual value to which \((1 + \frac{1}{p})^p\) approaches as \(p \to \infty\) is a number denoted by the letter \(e\). This number is irrational and is equal to 2.71828 to five decimal places. Therefore, the accurate formula for the value of the $100 investment after 4 years, with continuous compounding, is

\[100e^{0.32}\]

In general, if a sum \(p\) is compounded continuously at a nominal annual rate of interest of \(R\) percent,

\[\text{Value after } m \text{ years} = Pe^{\left(\frac{R}{100}\right)m}\]

**Example 1 (Investment)** An investment of $250 is compounded continuously at a nominal annual rate of interest of 7.5%. What will be the value of the investment after 6 years?

**Solution** We have \(P = 250\), \(R = 7.5\), \(m = 6\).

\[\text{Value after 6 years} = Pe^{\left(\frac{R}{100}\right)m} = 250e^{7.5 \times 6}\]

\[= 250e^{0.45} = 250 \times 1.5683 = 392.08 \text{ (dollars)}\]

2. **Present Value**

Let us suppose that by pursuing a certain business activity, a person expects to receive a certain sum of money, \(P\), at a time \(n\) years in the future. This future revenue \(P\) is less valuable than would be a revenue of the same amount received at the present time since, if the person received \(P\) now, it could be invested at interest, and it would be worth more than \(P\) in \(n\) years time. We are interested in finding the sum \(Q\) which, if received at the present time and invested for \(n\) years, would be worth the same as the future revenue \(P\) that person will receive.
Let us suppose that the interest rate on such an investment is equal to $R$ percent per annum, compounded annually. Then after $n$ years, the sum $Q$ would have the value equal to

$$Q \left(1 + \frac{R}{100}\right)^n.$$

Setting this equal to $P$, we obtain

$$Q \left(1 + \frac{R}{100}\right)^n = P, \quad Q = P/ \left(1 + \frac{R}{100}\right)^n = P \left(1 + \frac{R}{100}\right)^{-n}.$$

We call $Q$ the present value of the future revenue $P$, and call $R$ the discount rate at which the future revenue $P$ is discounted to the present value $Q$.

**Example 2. (Real Estate Sales Decision)** A real estate developer owns a piece of property that could be sold right away for $100,000. Alternatively, the property could be held for 5 years. During this time, the developer would spend $100,000 on developing it, it would then sell for $300,000. Assume the developer has to borrow from the bank for the $100,000 used for development at 12% interest per annum (compounded annually). If the discount rate is 10%, calculate the present value of this second alternative and hence decide which plan is best for the developer.

**Solution** The future value $P$ for the second plan is:

$$\begin{align*}
P &= (\text{money from the sale}) - (\text{cost of the loan}) \\
\text{money from the sale} &= $300,000 \\
\text{cost of the loan} &= $100,000(1.12)^5 = $197,382 \\
P &= $300,000 - $197,382 = $102,618
\end{align*}$$

Discounting back at a rate of 10%, we obtain a present value of $P$,

$$Q = P \left(1 + \frac{R}{100}\right)^{-5} = $102,618(1.1)^{-5} = \frac{$102,618}{1.61051} = $63,717.70.$$

Therefore, the first plan is much better.

**3. Exponential Functions**

A function of the type

$$y = a^x, \quad (a > 0, a \neq 1)$$

is called an exponential function, where $a$ is a constant. When $a > 1$, the function is called a growing exponential function, whereas when $a < 1$, it is called a decaying exponential function.
Example 3. \( y = 3^x \) is an example of a growing exponential function, while \( y = \left(\frac{1}{3}\right)^x \) is a decaying exponential function. Their graphs are shown below.

When \( a = e \approx 2.71828 \), the exponential function becomes \( y = e^x \), and it is called the natural exponential function. Can you use Excel to draw the graphs of \( y = e^x \), \( y = 2^x \), and \( y = 3^x \)?
11 Logarithms

Recall that the inverse of a function \( f(x) \) is obtained by solving the equation \( y = f(x) \) for \( x \), thus expressing \( x \) as a function of \( y : x = f^{-1}(y) \). We can consider the possibility of constructing the inverse of the exponential function \( a^x \). In order to do so, we must solve the equation \( y = a^x \) for \( x \). Though we can do this for concrete values of \( y \), we cannot express the inverse by a function we learned so far. So a new function must be invented for the inverse of \( a^x \). We write it in the form \( x = \log_a y \), and call it the **logarithm of \( y \) with base \( a \)**. Thus

\[
x = \log_a y \text{ if and only if } y = a^x
\]

As \( a^x \) is defined only for \( a > 0 \) and \( a \neq 1 \), we require the same for the base \( a \) in \( \log_a y \). Since \( y = a^x > 0 \) for any \( x \), we require \( y > 0 \) in \( \log_a y \), i.e., the domain of \( \log_a y \) is the set of all positive numbers. As \( x \) is usually used to denote the independent variable, the logarithm is usually given as \( \log_a x \).

**Example 1.** Consider the functions \( y = 3^x \) and \( y = \log_3 x \). By definition, \( y = 3^x \) is equivalent to \( x = \log_3 y \), and \( y = \log_3 x \) can be regarded as interchanging the positions of \( x \) and \( y \) in the equation \( x = \log_3 y \). Thus, the graph of \( y = \log_3 x \) is the same in shape and in size as that of \( y = 3^x \), except that it is the reflection of the graph of \( y = 3^x \) about the line \( y = x \).

**Properties of logarithms:**

1. \( \log_a 1 = 0 \)
2. \( \log_a a = 1 \)
3. \( \log_a (uv) = \log_a u + \log_a v \)
4. \( \log_a \left( \frac{u}{v} \right) = \log_a u - \log_a v \)
5. \( \log_a(u^n) = n \log_a u. \)

These properties follow directly from the corresponding properties of the exponential functions and the definition of logarithm.

1. \( a^0 = 1 \)
2. \( a^1 = a \)
3. \( a^\alpha \cdot a^\beta = a^{\alpha + \beta} \) (\( u = a^\alpha, v = a^\beta, \alpha = \log_a u, \beta = \log_a v \))
4. \( \frac{a^\alpha}{a^\beta} = a^{\alpha - \beta} \) (\( u = a^\alpha, v = a^\beta, \alpha = \log_a u, \beta = \log_a v \))
5. \( (a^\alpha)^n = a^{\alpha n} \) (\( u = a^\alpha, \alpha = \log_a u \))

Example 2 If \( x = \log_2 3 \), express the following quantities in terms of \( x \).

(a) \( \log_2 \left( \frac{1}{3} \right) \)
(b) \( \log_2 \left( \frac{2}{3} \right) \)
(c) \( \log_2 18 \)
(d) \( \log_2 \sqrt{\frac{27}{2}} \)

Solution

(a)
\[
\log_2 \left( \frac{1}{3} \right) = \log_2 1 - \log_2 3 = 0 - \log_2 3 = -x.
\]

(b)
\[
\log_2 \left( \frac{2}{3} \right) = \log_2 2 - \log_2 3 = 1 - \log_2 3 = 1 - x.
\]

(c)
\[
\log_2 18 = \log_2 (2 \times 9) = \log_2 2 + \log_2 9 = 1 + \log_2 9.
\]
\[
= 1 + \log_2 (3^2) = 1 + 2 \log_2 3 = 1 + 2x.
\]

(d)
\[
\log_2 \sqrt{\frac{27}{2}} = \log_2 \left( \frac{27}{2} \right)^{\frac{1}{2}} = \frac{1}{2} \log_2 \left( \frac{27}{2} \right) = \frac{1}{2} \left( \log_2 27 - \log_2 2 \right)\]
\[
= (\log_2 27 - 1) = \frac{1}{2} \left[ \log_2 (3^3) - 1 \right] = \frac{1}{2} \left[ 3 \log_2 3 - 1 \right] = \frac{1}{2} (3x - 1) = \frac{3x}{2} - \frac{1}{2}.
\]
Natural Logarithms

When the base is $e$, $\log_e x$ is called the natural logarithm, and it is denoted by $\ln x$. Thus $\log_e x = \ln x$ and $y = \ln x$ is equivalent to $e^y = x$.

The properties for $\log_a x$ carry to the special case $\ln x = \log_e x$ and they are:

1. $\ln 1 = 0$,
2. $\ln e = 1$
3. $\ln(uv) = \ln u + \ln v$
4. $\ln \left(\frac{u}{v}\right) = \ln u - \ln v$
5. $\ln(u^n) = n \ln u$.

Example 3. Solve the following equations for $x$.

(a) $2 \ln(x + 1) = \ln(2x + 2)$
(b) $\log_x(3 - 2x) = 2$.

Solution

(a) $2 \ln(x + 1) = \ln(x + 1)^2 = \ln(x^2 + 2x + 1)$. Therefore $\ln(x^2 + 2x + 1) = \ln(2x + 2)$. It follows, $x^2 + 2x + 1 = 2x + 2$, or $x^2 = 1$.

Hence $x = 1$ or $x = -1$. But if $x = -1$, then $x + 1 = 0$ and $\ln(x + 1)$ becomes $\ln 0$ which is not defined. Therefore only $x = 1$ is a solution.

(b) $\log_x(3 - 2x) = 2$ is equivalent to $x^2 = 3 - 2x$ which can be written as

$$x^2 + 2x - 3 = 0 \text{ or } (x - 1)(x - 3) = 0.$$ 

Hence $x = 1$ or $x = 3$. But $3 - 2x = -3 < 0$ when $x = 3$, which implies that $\log_2(3 - 2x)$ is not defined when $x = 3$. As $x$ is the base in $\log_x(3 - 2x), x = 1$ is also not acceptable. Thus there is no solution.

Common Logarithms

When the base is 10, $\log_{10} x$ is called the common logarithm, and is denoted by $\log x$. Hence $\log_{10} x = \log x$, and $y = \log x$ is equivalent to $10^y = x$.

Remark In some books, the notation $\log x$ is used to mean the natural logarithm of $x$. 
12 Applications of Logarithms

In many practical problems, one meets the problem of solving for $x$ from an equation of the type $a^x = b$, where $a$ and $b$ are positive constants. In some special cases, one can find $x$ by observation, for example, from $2^x = 4$, one finds $x = 2$. But in general, this method does not work. It is an application of the logarithm function that solves this problem. From $a^x = b$, we deduce

$$\ln(a^x) = \ln b$$

But $\ln(a^x) = x \ln a$. Therefore,

$$x \ln a = \ln b, \quad x = \frac{\ln b}{\ln a}$$

One could also use the common logarithm. Then the solution is

$$x = \frac{\log b}{\log a}.$$ 

Example 1 Solve the following equations for $x$.

(a) $3^x = 2^{2-x}$

Solution

(a) From $3^x = 2^{2-x}$, we obtain

$$\ln(3^x) = \ln(2^{2-x}), \quad x \ln 3 = (2 - x) \ln 2$$

$$x \ln 3 = 2 \ln 2 - x \ln 2,$$

$$x(\ln 3 + \ln 2) = 2 \ln 2, \quad x = \frac{2 \ln 2}{\ln 3 + \ln 2} \approx \frac{1.386}{1.792} \approx 0.773.$$

or

$$x = \frac{2 \ln 2}{\ln 3 + \ln 2} = \frac{\ln 2^2}{\ln(3 \times 2)} = \frac{\ln 4}{\ln 6} = \frac{1.386}{1.792} \approx 0.773.$$ 

(b) From $(2^x)^x = 25$, we deduce

$$\ln((2^x)^x) = 25, \quad x \ln(2^x) = 25$$

$$x(x \ln 2) = 25, \quad x^2 \ln 2 = 25$$

$$x^2 = \frac{25}{\ln 2}, \quad x = \pm \sqrt{\frac{25}{\ln 2}} = \pm \sqrt{\frac{25}{0.693}} = \pm 6.01$$

□
Example 2  The sum of $100 is invested at 6% compound interest per annum. How long does it take the investment to increase in value to $150?

Solution  The value of the investment after \( n \) years = \( 100 \left(1 + \frac{6}{100}\right)^n \)

\[= 100(1.06)^n.\]

We let this number equal 150, and find \( n \).

\[100(1.06)^n = 150,\]
\[(1.06)^n = \frac{150}{100} = 1.5\]
\[\ln(1.06)^n = \ln 1.5\]
\[n \ln 1.06 = \ln 1.5,\]
\[n = \frac{\ln 1.5}{\ln 1.06} = \frac{0.405}{0.058} = 6.98\]

Thus it takes almost 7 years for the investment to increase to $150.

Example 3.  What nominal rate of interest, when compounded continuously, gives the same growth over a whole year as a 10% annual rate of interest?

Solution.  A sum \( P \) invested at a nominal rate of interest \( R \) percent compounded continuously has a value \( Pe^{\left(\frac{R}{100}\right)} \) after 1 year. If invested at 10% per annum, the value after 1 year is \( P \left(1 + \frac{10}{100}\right) = P \times 1.1. \) To have the same growth rate, we should have

\[e^{\frac{R}{100}} = 1.01\]
\[\ln \left(e^{\frac{R}{100}}\right) = \ln 1.01, \quad \frac{R}{100} \ln e = \ln 1.01\]
\[\frac{R}{100} = \frac{\ln 1.01}{\ln e} = \ln 1.01\]
\[R = 100 \times \ln 1.01 = 9.53\]

Thus 10% interest compounded annually is equivalent to the annual growth provided by a 9.53% nominal rate of interest compounded continuously.

Example 4.  The population of a certain nation is given in millions by the formula

\[P = 15e^{0.02t}\]

where \( t \) is the time in years measured from 1990. When will the population reach 25 million?

Solution  Setting \( P = 25 \), we want to find \( t \).

\[25 = 15e^{0.02t}\]
\[
e^{0.02t} = \frac{25}{15} = 1.667
\]
\[
\ln(e^{0.02t}) = \ln 1.667
\]
\[
0.02t \ln e = \ln 1.667
\]
\[
0.02t = \ln 1.667
\]
\[
t = \frac{\ln 1.667}{0.02} = 25.5 \text{ (years)}
\]

Therefore, the population will reach 25 million in 25.5 years from 1990, that is in the middle of the year 2015.
13 Geometric Progressions and Savings Plans

Suppose $1000 is deposited with a bank that calculates interest at the rate of 10% compounded annually. The value of this investment (in dollars) at the end of \( n \) years is equal to

\[
1000 \left( 1 + \frac{10}{100} \right)^n = 1000(1.1)^n.
\]

Thus the values of the investment at the end of 0 years, 1 year, 2 years, 3 years, and so on, are

\[
1000, 1000(1.1), 1000(1.1)^2, 1000(1.1)^3, \ldots
\]

Note that the ratio of each term to its preceding term in the above sequence is the same, namely 1.1. Such sequences are called geometric progressions. A formal definition is given below.

**Definition** A sequence of terms is said to be a geometric progression (G.P.) if the ratio of each term to its preceding term is the same throughout. This ratio is called the **common ratio** of the G.P.

If \( a \) is the first term in a G.P., and \( r \) is the common ratio, then the G.P. has the form

\[
a, ar, ar^2, ar^3, \ldots
\]

Thus the \( n \)-th term is given by \( T_n = ar^{n-1} \).

**Example 1.** Find the fifth and \( n \)-th term of the G.P. 2, 6, 18, 54, \ldots.

**Solution** We have \( a = 2 \), \( r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3 \). Hence the fifth term is

\[
T_5 = ar^{5-1} = ar^4 = 2(3)^4 = 162.
\]

The \( n \)-th term is

\[
T_n = ar^{n-1} = 2(3)^{n-1}.
\]

**Example 2** (Depreciation) A machine is purchased for $10,000 and is depreciated annually at the rate of 20% of its declining value. Find an expression for the value after \( n \) years. If the ultimate scrap value is $3000, what is the effective life of the machine (i.e., the number of years until its depreciated value is less than its scrap value)?

**Solution** Since the value of the machine depreciates each year by 20% of its value at the beginning of the year, the value of the machine at the end of any year is 80% or 0.8 of its value at the beginning of the year. Thus the value of the machine at
the end of the \(n\)-th year is \(10000(0.8)^n\). To find the effective life of the machine, we let

\[
10000(0.8)^n = 3000
\]

and find \(n\). We have

\[
\begin{align*}
(0.8)^n &= \frac{3000}{10000} = 0.3 \\
\ln(0.8)^n &= \ln(0.3), n \ln(0.8) = \ln(0.3) \\
n &= \frac{\ln(0.8)}{-\ln(0.8)} = \frac{-1.204}{-0.223} = 5.4
\end{align*}
\]

Therefore the effective life of the machine is 5.4 years.

Let us use \(S_n\) to denote the sum of the first \(n\) terms of the G.P. given by

\[
a, ar, ar^2, ar^{n-2}, ar^{n-1}, \ldots,
\]

i.e.

\[
S_n = a + ar + \cdots + ar^{n-1}.
\]

Then

\[
-rS_n = -ar - ar^2 - \cdots - ar^{n-1} - ar^n
\]

Hence

\[
S_n - rS_n = a + (ar - ar) + (ar^2 - ar^2) + \cdots + (ar^{n-1} - ar^{n-1}) - ar^n
\]

\[
= a - ar^n = a(1 - r^n).
\]

i.e

\[
(1 - r)S_n = a(1 - r^n),
\]

\[
S_n = \frac{a(1 - r^n)}{1 - r}
\]

Example 3 (Savings Plan). Each year a person invests $1000 in a savings plan that pays interest at the fixed rate of 8% per annum. What is the value of this savings plan on the tenth anniversary of the first investment? (Including the current payment paid into the plan).

Solution The first $1000 has been invested for 10 years, so it has increased in value to

\[
1000 \left(1 + \frac{R}{100}\right)^{10} = 1000(1.08)^{10}.
\]

The second $1000 has increased in value to

\[
1000(1.08)^9
\]

as it has been invested for 9 years.
The third $1000 has value \(1000(1.08)^8\), \(\cdots\), and the 9th $1000 has value 1000. Thus the value of the savings plan is equal to

\[
1000 + 1000(1.08) + \cdots + 1000(1.08)^{10}
\]

This is the sum of 11 terms of the G.P. with \(a = 1000\), \(r = 1.08\). Hence

\[
S_{11} = \frac{a - r^{11}}{1 - r} = 1000 \cdot \frac{1 - 1.08^{11}}{1 - 1.08} = \frac{1000 \cdot 1.3316}{-0.08} = 16645.
\]

Thus the value is $16645.

\[\square\]

**Example 4** Every month Jane pays $100 into a savings plan that earns interest at 0.5% per month. Calculate the value of her savings immediately after making her \(n\)-th payment.

**Solution** The \(n\)-th payment is made after \(n - 1\) months of the first payment. So the value of the savings plan is the sum of

\[
100 \left(1 + \frac{0.5}{100}\right)^{n-1}, 100 \left(1 + \frac{0.5}{100}\right)^{n-2}, \cdots, 100 \left(1 + \frac{0.5}{100}\right), 100.
\]

This is the sum of \(n\) terms of the G.P.

\[100, 100(1.005), 100(1.005)^2, \cdots\]

Hence

\[
S_n = \frac{a\left(1 - r^n\right)}{1 - r} = 100 \frac{1 - (1.005)^n}{1 - 1.005} = \frac{100(1 - (1.005)^n)}{-0.005} = 20000(1.005^n - 1)
\]

\[\square\]
14 **Annuities and Amortization**

In a savings plan, one deposits to the bank at regular intervals of time, and at the end, one gets a sum of money from the bank. This sum of money is often called the future value of the savings plan.

Another situation is described by an annuity, where a person makes a deposit to the bank (or insurance company, etc) at the very beginning, and the bank (or insurance company, etc) pays that person a certain amount of money at regular time intervals until the deposit is used up. Let us look at the following example.

**Example 1** On his 65th birthday, Mr Hoskins wishes to purchase an annuity that will pay him $5000 per year for the next 10 years, the first payment to be made to him on his 66th birthday. His insurance company will give him an interest rate of 8% per annum on the investment. How much must he pay in order to purchase such an annuity?

**Solution** Mr Hoskins will get ten $5000 in the next 10 years and all his money comes from the deposit and the interest it earns.

To find the amount $A$ that he should pay initially, we imagine that the amount $A$ is divided into 10 parts, $A_1, A_2, \ldots, A_{10}$, in such a way that the value of $A_1$ after one year is the first $5000 he gets paid at his 66th birthday, the value of $A_2$ after 2 years is the second $5000 he gets paid, the value of $A_3$ after 3 years gives him the third $5000, \ldots$, the value of $A_{10}$ after 10 years is the last $5000 he gets paid.

Then we should have

\[
A_1(1 + 0.08) = 5000, \quad A_1 = 5000(1.08)^{-1} \\
A_2(1 + 0.08)^2 = 5000, \quad A_2 = 5000(1.08)^{-2} \\
A_3(1 + 0.08)^3 = 5000, \quad A_3 = 5000(1.08)^{-3} \\
\vdots \quad \vdots \\
A_{10}(1 + 0.08)^{10} = 5000, \quad A_{10} = 5000(1.08)^{-10}
\]

Then

\[
A = A_1 + A_2 + A_3 + \cdots + A_{10},
\]

which is the sum of the first 10 terms of a G.P. with $a = 5000(1.08)^{-1}$, $r = 1.08^{-1}$.

Therefore, 

\[
A = \frac{a(1 - r^{10})}{1 - r} = 5000(1.8)^{-1} \frac{1 - (1.08^{-1})^{10}}{1 - 1.08^{-1}} = 5000 \frac{1 - 1.08^{-10}}{1.08 - 1} = 33,5500 \text{(dollars)}.
\]
In general, suppose an annuity is purchased with a down payment \( A \), and the payments made to the annuitant are equal to \( P \) at regular intervals for \( n \) periods starting one period after the annuity is purchased, and assume the interest rate is \( R \) percent per period. Then

\[
A = P (1 + i)^{-1} + P (1 + i)^{-2} + \cdots + P (1 + i)^{-n} \quad (i = \frac{R}{100})
\]

\[
= a \frac{1 - r^n}{1 - r} \quad \text{(with } a = P (1 + i)^{-1}, r = (1 + i)^{-1})
\]

\[
= P (1 + i)^{-1} \frac{1 - (1 + i)^{-n}}{1 - (1 + i)^{-1}} = P \frac{1 - (1 + i)^{-n}}{(1 + i) - 1}
\]

\[
= \frac{P}{i} [1 - (1 + i)^{-n}]
\]

i.e.

\[
A = \frac{P}{i} [1 - (1 + i)^{-n}]
\]

**Example 2.** Mrs Josephs retires at the age of 63 and uses her life savings of $120,000 to purchase an annuity. The life insurance company gives an interest rate of 6% per annum, and they estimate that her life expectancy is 15 years. How much annuity (i.e., how big an annual pension) will she receive?

**Solution** We use the formula

\[
A = \frac{P}{i} [1 - (1 + i)^{-n}],
\]

where we know \( A = 120000 \), \( i = 6\% = 0.06 \) and \( n = 15 \).

Thus

\[
120000 = \frac{P}{0.06} [1 - (1 + 0.06)^{-15}]
\]

\[
= \frac{P}{0.06} [1 - 1.06^{-15}]
\]

\[
= P \times 9.712249
\]

\[
P = \frac{120000}{9.712249} = 12355.53 \text{ (dollars)}
\]

Hence Mrs Josephs will receive an annual pension of $12,355.53.

When a debt is repaid by regular payments over a period of time, we say that the debt is amortized. Examples of this are car loans, mortgages, etc.
Mathematically speaking, the amortization of a debt presents exactly the same problem as paying an annuity. With an annuity, we can view the annuitant as lending a certain amount $A$ to the insurance company; the company then repays this loan by $n$ regular payments of amount $P$ each.

**Example 3.** A certain student borrowed $8000 from the bank to buy a car. The interest rate is 8% per annum and the student repays in single installments at the end of each year. How much must the student pay each year to repay the loan in 5 years?

**Solution** We use the formula

$$A = \frac{P}{i} \left[ 1 - (1 + i)^{-n} \right]$$

where $a = 8000$, $i = 8\% = 0.08$, $n = 5$

Hence

$$8000 = \frac{P}{0.08} \left[ 1 - (1 + 0.08)^{-5} \right]$$

$$= \frac{P}{0.08} (1 - 1.08^{-5})$$

$$= P \times 3.99271$$

$$P = \frac{8000}{3.99271} = 2003.65 \text{ (dollars).}$$

The student must repay $2003.65 each year.

**Example 4** A married couple have a combined income of $45,000. Their mortgage company will allow them to borrow up to an amount at which the repayments are one-third of their income. If the interest rate is 1.2% per month, amortized over 25 years, how much can they borrow?

**Solution** Again we use the formula

$$A = \frac{P}{i} \left[ 1 - (1 + i)^{-n} \right]$$

where $P = \left( \frac{1}{3} \times 54000 \right) \div 12 = 1250$

$$i = 1.2\% = 0.012$$

$$n = 12 \times 25 = 300$$

Hence

$$A = \frac{1250}{0.012} \left[ 1 - (1 + 0.012)^{-300} \right]$$

$$= \frac{1250}{0.012} (1 - 1.012^{-300})$$

$$= 101,258.80 \text{ (dollars).}$$

Therefore, they can borrow $101,258.80.$
15 Matrices

A firm produces four products, $A$, $B$, $C$ and $D$. The costs of producing each of these four products consist of the use of material $X$, the use of material $Y$ and labor. The table below shows a sample of such costs for each product.

<table>
<thead>
<tr>
<th>Product</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units of $X$</td>
<td>250</td>
<td>300</td>
<td>170</td>
<td>200</td>
</tr>
<tr>
<td>Units of $Y$</td>
<td>160</td>
<td>230</td>
<td>75</td>
<td>10</td>
</tr>
<tr>
<td>Units of labor</td>
<td>80</td>
<td>85</td>
<td>120</td>
<td>100</td>
</tr>
</tbody>
</table>

Observe that the data in this table naturally form a rectangular array. If the headings are removed, we obtain the following rectangular array of numbers.

\[
\begin{bmatrix}
250 & 300 & 170 & 200 \\
160 & 230 & 75 & 10 \\
80 & 85 & 120 & 100
\end{bmatrix}
\]

This array is an example of a matrix.

**Definition.** A **matrix** is a rectangular array of real numbers, which is enclosed in large brackets. Matrices are generally denoted by capital letters $A$, $B$, etc.

Some examples of matrices are

\[
A = \begin{bmatrix}
2 & -3 & 7 \\
1 & 0 & 4
\end{bmatrix}, \quad B = \begin{bmatrix}
3 & 4 \\
7 & 8 \\
1 & 2
\end{bmatrix}, \quad C = \begin{bmatrix}
6 \\
7 \\
1 \\
4
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 3 \\
2 & 5
\end{bmatrix}.
\]

The real numbers in a matrix are called **entries** or **elements** of the matrix. The elements in any horizontal line form a **row** and those in any vertical line form a **column** of the matrix. For example, the matrix $B$ above has 3 rows and 2 columns, whereas $C$ has 4 rows and 1 column.

If a matrix has $m$ rows and $n$ columns, then it is said to be of **size** $m \times n$ (read $m$ by $n$). Of the matrices given above, $A$ is a $2 \times 3$ matrix, $B$ is a $3 \times 2$ matrix, $C$ is a $4 \times 1$ matrix, and $D$ is a $2 \times 2$ matrix.
To denote a general $m \times n$ matrix, we usually use

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}$$

This is also denoted by $A = [a_{ij}]_{m \times n}$.

If a matrix has all its elements zero, it is called a **zero matrix**. Thus the following is the zero matrix of size $2 \times 3$

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

**Definition** Two matrices $A$ and $B$ are said to be **equal** if
(1) they are of the same size, and
(2) their corresponding elements are equal.

For example, let

$$A = \begin{bmatrix}
2 & x & 3 \\
y & -1 & 4
\end{bmatrix} \text{ and } B = \begin{bmatrix}
a & 1 & b \\
b & 6 & 4
\end{bmatrix}.$$  

Clearly, $A$ and $B$ are of the same size, and $A = B$ if and only if $a = 2, x = 1, b = 3, y = 6$.

**Operations of Matrices**

1. **Scalar Multiplication:**
   If $c$ is a scalar (i.e., a number), and $A = [a_{ij}]_{m \times n}$ a matrix, then
   $$cA = [ca_{ij}]_{m \times n}$$
   i.e. $cA$ is obtained by multiplying each element of $A$ by $c$.

2. **Addition and Subtraction:**
   If $A$ and $B$ are matrices of the same size, say they are both of size $m \times n$, then $A + B$ and $A - B$ are defined. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then
   $$A + B = [a_{ij} + b_{ij}]_{m \times n}, \ A - B = [a_{ij} - b_{ij}]_{m \times n}.$$  
   i.e., $A + B$ is obtained by adding the corresponding elements of $A$ and $B$, whereas $A - B$ is obtained by subtracting the corresponding elements of $B$ from $A$.  

Example 1  Let \( A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \), \( C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \).

Determine which of the following operations are defined, and when it is defined, perform the operation.

\( A + B, 3A - B, B + C, 2C \)

Solution  Since \( A \) and \( B \) are of the same size, \( A + B \) is defined, and

\[
A + B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}
\]

\( 3A = 3 \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 \\ 0 & 3 & 6 \\ 3 & 6 & 9 \end{bmatrix} \)

\( 3A \) and \( B \) are of the same size, therefore \( 3A - B \) is defined, and

\[
3A - B = \begin{bmatrix} 6 & 3 & 0 \\ 0 & 3 & 6 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 0 & 3 & 5 \\ 3 & 5 & 9 \end{bmatrix}
\]

\( B \) and \( C \) are not of the same size, therefore \( B + C \) is not defined.

\( 2C = 2 \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 2 & 6 \end{bmatrix} \).

Example 2. Let \( A = \begin{bmatrix} x^2 & y - 1 \\ u & v^3 + 1 \end{bmatrix} \), \( B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} \).

Suppose \( A = B \). Find \( x, y, u \) and \( v \).

Solution  Since \( A = B \), their corresponding elements are equal. Hence we have \( x^2 = 4, y - 1 = 6, u = 3, v^3 + 1 = 2 \). It follows \( x = \pm \sqrt{4} = \pm 2, y = 6 + 1 = 7, u = 3, v^2 = 2 - 1 = 1, v = \pm \sqrt{1} = \pm 1 \).
16 Multiplication of Matrices

In the last lecture, we have learned the scalar multiplication, addition, and subtraction of matrices. These operations are very similar to the corresponding operations for real numbers. In this lecture, we discuss matrix multiplications. We will see that this is very different from multiplications of real numbers.

Let us look at a situation that matrix multiplication arises. Suppose a firm manufactures a product using three different amounts of three inputs, $P$, $Q$ and $R$ (they could be materials or labour, for example). Let the number of units of these inputs used for each unit of the product be given by the following row matrix.

\[
A = \begin{bmatrix} P & Q & R \\ 3 & 2 & 4 \end{bmatrix}
\]

Then let the cost per unit of each of the three inputs be given by the following column matrix

\[
B = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}
\]

Then the total cost of producing one unit of the product is

\[
3 \times 10 + 2 \times 8 + 4 \times 6 = 70.
\]

We refer to this number as the product of the row matrix $A$ and the column matrix $B$, written $AB$. Observe that in forming $AB$, the first elements of $A$ and $B$ are multiplied together, the second elements are multiplied together, the third elements are multiplied together, and then these three products are added.

This method of forming products applies to row and column matrices of any size.

**Definition** Let $C$ be a $1 \times n$ row matrix and $D$ be an $n \times 1$ column matrix. Then the product $CD$ is obtained by calculating the products of corresponding elements in $C$ and $D$ and then adding all $n$ of these products. Note: We do not use $DC$ to denote this product of $C$ and $D$.

**Example 1** Given $K = \begin{bmatrix} 2 & 5 \end{bmatrix}$, $L = \begin{bmatrix} 1 & -2 & -3 & 2 \end{bmatrix}$, $M = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $N = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 4 \end{bmatrix}$.

Then

\[
KM = (2)(-3) + (5)(2) = -6 + 10 = 4.
\]
\[ LN = (1)(2) + (-2)(5) + (-3)(-3) + (2)(4) = 2 - 10 + 9 + 8 = 9 \]

Note: The row and column matrices must have the same number of elements in order to make a product. In example 1, the products \(LM\) and \(KN\) are not defined.

In more general situations, one needs to find the product of two matrices \(A\) and \(B\), but \(A\) may not be a row matrix, and \(B\) may not be a column matrix. The general definition is given below.

**Definition** If \(A = [a_{ij}]\) is an \(m \times n\) matrix and \(B = [b_{ij}]\) is an \(n \times p\) matrix, then the product \(AB\) is an \(m \times p\) matrix \(C = [c_{ij}]\), where the \(ij\)-th element \(c_{ij}\) of \(C\) is obtained by multiplying the \(i\)-th row of \(A\) and the \(j\)-th column of \(B\). Note that the number of columns in \(A\) and the number of rows in \(B\) must be the same in this definition.

**Example 2** Let
\[
A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}
\]

Find \(AB\) and \(BA\) if they exist.

**Solution** Here \(A\) is \(2 \times 2\) and \(B\) is \(2 \times 3\). Since the number of columns in \(A\) is equal to the number of rows in \(B\), the product \(AB\) is defined. It is of size \(2 \times 3\). If \(C = AB\), then we can write \(C\) in the form
\[
C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}
\]

The element \(c_{ij}\) is found by multiplying the \(i\)-th row of \(A\) and the \(j\)-th column of \(B\). For example,
\[
c_{12} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (2)(1) + (3)(-3) = 2 - 9 = -7
\]

In full, we have
\[
AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} (2)(3) + (3)(2) & (2)(1) + (3)(-3) & (2)(0) + (3)(4) \\ (4)(3) + (1)(2) & (4)(1) + (1)(-3) & (4)(0) + (1)(4) \end{bmatrix} = \begin{bmatrix} 12 & -7 & 12 \\ 14 & 1 & 4 \end{bmatrix}.
\]

The product \(BA\) is not defined because the number of columns in the left matrix \(B\) is not equal to the number of rows in the right matrix \(A\). \(\square\)
Example 3  Given \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix} \) and \( B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \), find \( AB \) and \( BA \).

Solution  Here \( A \) and \( B \) are both of size \( 3 \times 3 \). Thus \( AB \) and \( BA \) are both defined and both have size \( 3 \times 3 \). We have the following

\[
AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} 
= \begin{bmatrix} 7 & 14 & 10 \\ 13 & 32 & 25 \\ 3 & 16 & 13 \end{bmatrix}
\]

\[
BA = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix} 
= \begin{bmatrix} 6 & 3 & 8 \\ 13 & 17 & 25 \\ 17 & 19 & 29 \end{bmatrix}
\]

Let us note that \( AB \neq BA \), even though both products are defined. \( \Box \)
17 Multiplication of Matrices (continued)

Recall that if \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times p \) matrix, then \( AB \) is defined and is of size \( m \times p \).

If \( A \) and \( B \) are as above, and \( C \) is a \( p \times q \) matrix, then the products \( AB, BC, (AB)C \) and \( A(BC) \) are all defined. It can be proved that

\[
(AB)C = A(BC).
\]

Therefore we usually omit the brackets and write simply \( ABC \). Note that \( AB \) is of size \( m \times q \), \( BC \) is of size \( n \times q \) and \( ABC \) is of size \( m \times q \).

Recall that a matrix with all its elements 0 is called a zero matrix. If \( A \) is an \( m \times n \) matrix and 0 is the zero matrix of size \( m \times n \), then it follows easily from the definition of matrix addition that

\[
A + 0 = 0 + A = A.
\]

Thus in matrix addition the zero matrix plays the role the number 0 plays in addition of real numbers.

There is also a matrix which plays the role as the number 1 in multiplications of real numbers. This is the identity matrix defined below.

**Definition** A square matrix is called an identity matrix if all the elements on its diagonal are equal to 1 and all the other elements are equal to zero. Here by a square matrix we mean a matrix with the same number of rows and columns, and diagonal elements of a square matrix are the elements \( a_{ij} \) with \( i = j \).

For example,

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

are the identity matrices of size \( 2 \times 2 \) and \( 3 \times 3 \), respectively.

The identity matrix is usually denoted by \( I \) when its size is understood without ambiguity. To emphasize that an identity matrix has size \( n \times n \), we usually write \( I_{n \times n} \).

**Example 1.** Let

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]
Find $AI$ and $IA$, where $I$ denote the $2 \times 2$ identity matrix.

Solution

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a(1) + b(0) & a(0) + b(1) \\ c(1) + d(0) & c(0) + d(1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (1)a + (0)c & (1)b + (0)d \\ (0)a + (1)c & (0)b + (1)d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$  

Thus $AI = IA = A$. \qed

It can be easily checked that the identity

$$AI = IA = A$$

is true for any square matrix $A$, where $I$ is the identity matrix the same size as $A$. Moreover, if $I$ is $n \times n$, $A$ is $m \times n$ and $B$ is $n \times p$, then $AI = A, IB = B$.

Example 2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Find $AB$ and $BA$.

Solution

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1)(0) + (0)(1) & (1)(0) + (0)(0) \\ (0)(0) + (0)(1) & (0)(0) + (0)(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (0)(1) + (0)(0) & (0)(0) + (0)(0) \\ (1)(1) + (0)(0) & (1)(0) + (0)(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = B.$$  

Let us note that in Example 2, neither $A$ nor $B$ is a zero matrix, but $AB = 0$, the zero matrix. We know that $BI = B$ where $I$ is the $2 \times 2$ matrix, and clearly $A \neq I$, but we have $BA = B$. These are properties which are very different from that of real number multiplications.
By using the idea of matrix multiplications, we can write systems of linear equations in the form of matrix equations. Consider, for example, the system

\[
\begin{align*}
2x - 3y &= 7 \\
4x + y &= 21
\end{align*}
\]

This is equivalent to

\[
\begin{bmatrix}
2x - 3y \\
4x + y
\end{bmatrix} = \begin{bmatrix}
7 \\
21
\end{bmatrix}
\]

But it is easily checked that

\[
\begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
2x - 3y \\
4x + y
\end{bmatrix}
\]

Therefore the original system can be written as

\[
\begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
7 \\
21
\end{bmatrix}
\]

If we define matrices \(A, B\) and \(X\) as

\[
A = \begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
7 \\
21
\end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix}
x \\
y
\end{bmatrix},
\]

then this matrix equation can be written as

\[
AX = B.
\]

The advantage of writing an equation system in matrix form is the following. If we can find a matrix \(C\) such that \(CA = I\), the identity matrix, then from \(AX = B\), we obtain

\[
C(AX) = CB, (CA)X = CB, IX = CB, X = CB.
\]

For example, in the case \(A = \begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix}\), we can check that \(C = \begin{bmatrix}
\frac{1}{14} & \frac{3}{14} \\
-\frac{2}{7} & \frac{1}{7}
\end{bmatrix}\) satisfies \(CA = I\)

Hence \(X = CB = \begin{bmatrix}
\frac{1}{14} & \frac{3}{14} \\
-\frac{2}{7} & \frac{1}{7}
\end{bmatrix} \begin{bmatrix}
7 \\
21
\end{bmatrix} = \begin{bmatrix}
\left(\frac{1}{14}\right)(7) + \left(\frac{3}{14}\right)(21) \\
\left(-\frac{2}{7}\right)(7) + \left(\frac{1}{7}\right)(21)
\end{bmatrix} = \begin{bmatrix}
5 \\
1
\end{bmatrix}
\]

That is \(x = 5, y = 1\) is a solution of the original system.

**Example 3.** Express the following system of equations in matrix form.

\[
\begin{align*}
2x + 3y + 4z &= 7 \\
4y - 5z &= 2 \\
-2x + 3z &= -6
\end{align*}
\]
Solution  The system can be rewritten as

\[
\begin{align*}
2x + 3y + 4z &= 7 \\
0x + 4y - 5z &= 2 \\
-2x + 0y + 3z &= -6
\end{align*}
\]

If we define

\[
A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & -5 \\ -2 & 0 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix},
\]

then the system is equivalent to

\[AX = B.\]
18 The Inverse of a Matrix and Determinant

In the last lecture, we mentioned that systems of linear equations can be written as matrix equations of the form

\[ AX = B \]

where \( X \) is the column matrix containing all the unknowns. For example,

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 0 \\
1 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
1 \\
3
\end{bmatrix}
\]

is equivalent to a system of linear equations with three unknowns \( x, y \) and \( z \).

If we know a matrix \( C \) such that \( CA = I \), the identity matrix, then \( X = CAX = CB \). In other words, once such a matrix \( C \) can be found, then the solutions can be found through a simple calculation of the matrix multiplication \( CB \). However, finding such a matrix \( C \) is usually not easy. Nevertheless, this way of solving equation systems is particularly useful when one tries to use computers to find the solutions, and we will discuss this in detail later.

Finding a matrix \( C \) such that \( CA = I \) is closely related to finding the inverse of a matrix. Indeed, if \( A \) is a square matrix, such a matrix \( C \) is actually the inverse of \( A \). A formal definition is given below.

Definition Let \( A \) be an \( n \times n \) square matrix. If there is an \( n \times n \) matrix \( B \) such that \( AB = I \) and \( BA = I \), then we say \( A \) is invertible or nonsingular, and call \( B \) the inverse of \( A \), and usually denote it by \( B = A^{-1} \).

Example 1. Show that \( A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \) has no inverse.

Proof Let \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be an arbitrary \( 2 \times 2 \) matrix. Then

\[
AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 0 & 0 \end{bmatrix} \neq I.
\]

Hence \( A \) has no inverse. \( \square \)

If a square matrix has no inverse, then it is called a singular matrix. Note that we do not talk about the inverses for non-square matrices.

Example 2. Find the inverse of \( A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \).
**Solution.** Let \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be the inverse of \( A \). Then

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a + b \\ c & 2c + d \end{bmatrix}
\]

Therefore, \( a = 1, 2a + b = 0, c = 0, 2c + d = 1 \).

It follows, \( b = -2a = -2, d = 1 - 2c = 1 \). Thus

\[
B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.
\]

Checking \( AB \), we have

\[
AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\]

Hence \( B \) is indeed the inverse of \( A \). \( \square \)

Note that in Example 2, we only used \( BA = I \) to find \( B \), and it turns out that this \( B \) also satisfies \( AB = I \). This is no coincident. It can be proved that if the \( n \times n \) matrices \( A \) and \( B \) satisfy \( BA = I \), then we must have \( AB = I \).

Moreover, if \( A, B, C \) are all \( n \times n \) matrices, and satisfy

\[
AB = I, AC = I
\]

then we must have \( B = C \). In other words, the inverse of a matrix \( A \) is unique. This can be proved as follows.

From \( AB = I \) and the property mentioned above, we know \( BA = I \). Hence, using \( AC = I \), we obtain

\[
B(AC) = BI = B \\
(BA)C = B, IC = B, C = B
\]

Given an \( n \times n \) matrix \( A \), it is in general not easy to find the inverse of \( A \). Indeed, one needs to know before hand whether \( A \) has an inverse. Even this latter problem is not easy. If the size of a matrix \( A \) is not too big, there is a row reduction method which is very effective in finding the inverse of \( A \), or finding whether \( A \) has an inverse manually. However, in this unit, we are not going to use this method. Instead, we will learn to use Excel to determine whether \( A \) has an inverse, and when the inverse exists, we will use Excel to find it. The details are contained in the Practical Class Work Book.
One effective method to determine whether a given $n \times n$ matrix $A$ has an inverse is to calculate its determinant, denoted by $\det(A)$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}.$$ 

When $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$ 

For $A$ of the size bigger than $3 \times 3$, its determinant is defined in the fashion of the above definition for $3 \times 3$ matrices, but we will not go into the details. The interested reader is referred to the reference book, where more material on the topic of inverses and determinants can be found.

Now let us see how the determinant can be used to determine whether a matrix has an inverse.

**Theorem** An $n \times n$ matrix has an inverse if and only if its determinant is nonzero.

The proof of this theorem is highly nontrivial and is beyond the scope of this unit. Therefore we are only required to know the conclusion.

**Example 3.** Determine which of the following matrices has an inverse.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

**Solution** $\det(A) = 1 \times 4 - 2 \times 2 = 4 - 4 = 0$. Hence $A$ has no inverse. $\det(B) = 1 \times 0 - 0 \times 0 = 0$. Hence $B$ has no inverse. $\det(C) = 2 \times 4 - 1 \times 3 = 8 - 3 = 5 \neq 0$. Hence $C$ has an inverse. 

□
19 Solutions of Linear Systems

A general system of $m$ linear equations involving $n$ variables, denoted by $x_1, x_2, \cdots x_n$, can be expressed in the following form

$$
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1
$$
$$
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2
$$
$$
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3
$$
$$
\vdots
$$
$$
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
$$

If we denote by $A = [a_{ij}]_{m \times n}$ the $m \times n$ matrix consisting of the coefficients of the $n$ variables $x_1, \cdots, x_n$, in the $m$ equations, and

$$
X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},
B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},
$$

then the system can be rewritten as a matrix equation

$$
AX = B
$$

It can be proved that a given system of linear equations can have no solution, or a unique solution, or infinitely many solutions. But it can never happen that such a system has exactly two, or three, etc. solutions.

**Example 1.** The system

$$
x + 2y = 3
$$
$$
3x + 6y = 2
$$

has no solution, since if $x, y$ satisfy both equations, then we multiply the first equation by 3 and obtain

$$
3(x + 2y) = 9, \quad 3x + 6y = 9
$$

which is in contradiction to the second equation in the system.

**Example 2.** The system

$$
x + 2y = 3
$$
$$
x + y = 1
$$
has a unique solution \( x = -1, y = 2 \). Indeed, subtracting the second equation from the first, we obtain
\[
(x + 2y) - (x + y) = 301, \ y = 2.
\]
Substituting \( y = 2 \) into either the first or the second equation, we obtain \( x = -1 \).
\( \square \)

**Example 3.** The System

\[
\begin{align*}
x + 2y &= 3 \\
3x + 6y &= 6
\end{align*}
\]

has infinitely many solutions. Indeed, let \( t \) be any real number, and \( y = t, x = 3 - 2t \).

Then
\[
\begin{align*}
x + 2y &= (3 - 2t) + 2t = 3 \\
3x + 6y &= 3(3 - 2t) + 6t = 9.
\end{align*}
\]

Hence, for any \( t, x = 3 - 2t, y = t \) is a solution to the system.

\( \square \)

Let us now return to the general equation

\[
AX = B
\]

where \( A = [a_{ij}]_{m \times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \).

If the number of equations, \( m \), and the number of unknowns, \( n \), are equal, then \( A \) becomes an \( n \times n \) matrix and hence we can talk about its inverse \( A^{-1} \). We know from the last lecture that \( A^{-1} \) exists if and only if its determinant, \( \det(A) \), is not zero. In such a case,

\[
\begin{align*}
A^{-1}(Ax) &= A^{-1}B \\
(A^{-1}A)X &= A^{-1}B \\
IX &= A^{-1}B \\
X &= A^{-1}B
\end{align*}
\]

That is, the system has a unique solution which is given by
\[
X = A^{-1}B
\]

**Example 4** Find the solutions to the following system
\begin{align*}
2x - 3y + 4z &= 13 \\
x + y + 2z &= 4 \\
3x + 5y - z &= -4
\end{align*}

**Solution** Let \( A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix} \). Then,

\[
\det(A) = -35 \quad \text{(By Excel)}
\]

and again by Excel

\[
A^{-1} = \begin{bmatrix} 0.314285714 & -0.484714286 & 0.285714286 \\ -0.2 & 0.4 & 0 \\ -0.057142857 & 0.542857143 & -0.142857143 \end{bmatrix}
\]

and

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 13 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.
\]

\[\square\]

**Example 5.** Two products \( A \) and \( B \) are competitive. The demands \( x_A \) and \( x_B \) for these products are related to their prices \( P_A \) and \( P_B \) according to the demand equations

\[x_A = 17 - 2P_A + 0.5P_B, \quad x_B = 20 - 3P_B + 0.5P_A\]

The supply equations are

\[P_A = 2 + x_A + \frac{1}{3}x_B, \quad P_B = 2 + \frac{1}{2}x_B + \frac{1}{4}x_A\]

For market equilibrium, all four equations must be satisfied (i.e. supply equals demand). Find the equilibrium values of \( x_A, x_B, P_A \) and \( P_B \).

**Solution** The four equations can be rewritten into the standard form:

\begin{align*}
x_A + 0x_B + 2P_A - 0.5P_B &= 17 \\
0x_A + x_B - 0.5P_A + 3P_B &= 20 \\
x_A + \frac{1}{3}x_B - P_A + 0P_B &= -2 \\
\frac{1}{4}x_A + \frac{1}{2}x_B + 0P_A - P_B &= -2
\end{align*}
Thus the coefficient matrix is

\[
C = \begin{bmatrix}
1 & 0 & 2 & -0.5 \\
0 & 1 & -0.5 & 3 \\
1 & \frac{1}{3} & -1 & 0 \\
\frac{1}{4} & \frac{1}{2} & 0 & -1
\end{bmatrix}
\]

\[\det(C) = 6.6 \neq 0\]

\[
C^{-1} = \begin{bmatrix}
0.353312303 & -0.063091483 & 0.738170347 & -0.3659306 \\
-0.03785489 & 0.43533123 & -0.293375394 & 1.324921136 \\
0.340694006 & 0.082018927 & -0.359621451 & 0.075709779 \\
0.069400631 & 0.201892744 & 0.03785489 & -0.42902208
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
x_A \\
x_B \\
P_A \\
P_B
\end{bmatrix}
= C^{-1} \begin{bmatrix}
17 \\
20 \\
-2 \\
-2
\end{bmatrix}
= \begin{bmatrix}
4 \\
6 \\
8 \\
6
\end{bmatrix}
\]

i.e., \( x_A = 4, x_B = 6, P_A = 8, P_B = 6 \)
20 The Input-Output Model

The input-output model was first introduced in the late forties by Leontief, the recipient of a 1973 Nobel Prize, in a study of the U.S. economy. The main feature of this model is that it incorporates the interactions between different industries or sectors which make up the economy. The aim of the model is to allow economists to forecast the future production levels of each industry in order to meet future demands for the various products. Such forecasting is complicated as a change in the demand for one product can induce a change in production levels of many industries. For example, an increase in the demand for cars leads not only to an increase in the production level of automobile manufacturers, but also in the levels of many other industries in the economy, such as the steel industry, the rubber industry, and so on. In Leontief’s original model, he divided the U.S. economy into 500 interacting sectors of this type.

In order to discuss the model in the simplest possible terms, we consider a hypothetical economy with only two industries $P$ and $Q$. The interaction between $P$ and $Q$ are described by the following table

<table>
<thead>
<tr>
<th>Industry</th>
<th>Industry $P$ Inputs</th>
<th>Industry $Q$ Inputs</th>
<th>Consumer Demands</th>
<th>Total Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Industry P</td>
<td>60</td>
<td>64</td>
<td>76</td>
<td>200</td>
</tr>
<tr>
<td>Industry Q</td>
<td>100</td>
<td>48</td>
<td>12</td>
<td>160</td>
</tr>
<tr>
<td>Labor</td>
<td>40</td>
<td>48</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first two columns in this table give the inputs of $P$ and $Q$, measured in suitable units (e.g., millions of dollars). From the first column, we see that $P$ uses 60 units of its own product, 100 units of $Q$’s product and 40 units of labor. Similarly, from the second column, $Q$ uses 64 units of $P$’s product, 48 units of its own product and 48 units of labor. Totalling the columns, we see that the total inputs for $P$ is 200 units, and that for $Q$ is 160 units.

Now consider the first two rows in the table. The first row shows how the outputs of $P$ are used: 60 units for $P$ itself, 64 units for $Q$ and 76 units for consumers, totalling 200, which is the same as $P$’s total inputs. Similarly, the outputs of $Q$ are used as follows: 100 units for $P$, 48 units for $Q$ itself, and 12 units for consumers, totalling 160 units, the same as $Q$’s total inputs.

Suppose that market research predicts that in 5 years, the consumer demand for $P$ will decrease from 76 to 70 units, whereas for $Q$, it will increase considerably from
12 to 60 units. Then how much each industry should adjust its production level in order to meet these projected consumer demands?

Let us suppose that in order to meet this new demands, \( P \) must produce \( x_1 \) units and \( Q \) must produce \( X_2 \) units.

From the first column in the table we see that to produce 200 units, \( P \) uses 60 units of its own produce and 100 units of \( Q \)’s product. Thus to produce \( X_1 \) units, \( P \) must use
\[
\frac{60}{200} x_1 \quad \text{units of its own product and} \\
\frac{100}{200} x_1 \quad \text{units of } Q \text{’s product.}
\]

Similarly, using the second column in the table we see that to produce \( x_2 \) units, \( Q \) must use
\[
\frac{64}{160} x_2 \quad \text{units of } P \text{’s product and} \\
\frac{48}{160} x_2 \quad \text{units of its own product.}
\]

Thus we have a new table describing the interaction of \( P \) and \( Q \) under the new demands.

<table>
<thead>
<tr>
<th>( P ) outputs</th>
<th>( P ) inputs</th>
<th>( Q ) inputs</th>
<th>Consumer demands</th>
<th>Total outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( \frac{60}{200} x_1 )</td>
<td>( \frac{64}{160} x_2 )</td>
<td>70</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( \frac{100}{200} x_1 )</td>
<td>( \frac{48}{160} x_2 )</td>
<td>60</td>
<td>( x_2 )</td>
</tr>
</tbody>
</table>

Thus we have

\[
\begin{align*}
\frac{60}{200} x_1 + \frac{64}{160} x_2 + 70 \\
\frac{100}{200} x_1 + \frac{48}{160} x_2 + 60
\end{align*}
\]

These two equations can be written in matrix form as
\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{60}{200} & \frac{64}{160} \\ \frac{100}{200} & \frac{48}{160} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 70 \\ 60 \end{bmatrix}
\]

Thus
\[
X = AX + D,
\]
where \( X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is called the output matrix, \( D = \begin{bmatrix} 70 \\ 60 \end{bmatrix} \) is called the demand matrix, and \( A = \begin{bmatrix} 60 & 64 \\ 100 & 48 \\ 200 & 160 \end{bmatrix} \) is called the input-output matrix.

From \( X = AX + D \), we obtain

\[
X - AX = D, (I - A)X = D.
\]

If \( (I - A)^{-1} \) exists, then

\[
X = (I - A)^{-1}D.
\]

In our case here,

\[
I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 60 & 64 \\ 100 & 48 \\ 200 & 160 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.3 & 0.4 \\ 0.5 & 0.3 \end{bmatrix} - \begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix}
\]

\[
(I - A)^{-1} = \frac{1}{29} \begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix},
\]

\[
X = (I - A)^{-1}D = \frac{1}{29} \begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix} \begin{bmatrix} 70 \\ 60 \end{bmatrix} = \begin{bmatrix} 251.7 \\ 265.5 \end{bmatrix}
\]

Thus \( P \) must produce 251.7 units and \( Q \) should produce 265.5 units.

**Example 1.** Suppose that in a hypothetical economy with two industries \( A \) and \( B \), the interaction between \( A \) and \( B \) is as shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>demands</th>
<th>total outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>240</td>
<td>750</td>
<td>210</td>
<td>1200</td>
</tr>
<tr>
<td>B</td>
<td>720</td>
<td>450</td>
<td>330</td>
<td>1500</td>
</tr>
<tr>
<td>Primary Inputs</td>
<td>240</td>
<td>300</td>
<td></td>
<td>1200</td>
</tr>
</tbody>
</table>

(a) Find the input-output matrix \( A \).

(b) Determine the output matrix if the demands change to 312 units for industry \( A \) and 299 units for industry \( B \).
(c) What will then be the new primary inputs for \(A\) and \(B\)?

**Solution**

(a) \(A = \begin{bmatrix} \frac{240}{1200} & \frac{750}{1500} \\ \frac{720}{1200} & \frac{300}{1500} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.5 \\ 0.6 & 0.3 \end{bmatrix} \)

(b)

\[ D = \begin{bmatrix} 312 \\ 299 \end{bmatrix} \]

\[ X = (I - A)^{-1}D = \frac{5}{13} \begin{bmatrix} 7 & 5 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 312 \\ 299 \end{bmatrix} = \begin{bmatrix} 1415 \\ 1640 \end{bmatrix} \]

(c) New primary inputs for \(A\) = \(\frac{240}{1200} \times \) (new outputs of \(A\))

\[ = 0.2 \times 1415 = 283 \]

New primary inputs for \(B\) = \(\frac{300}{1500} \times \) (new outputs of \(B\))

\[ = 0.2 \times 1640 = 328 \]

\[ \square \]

Note that the given table gives the proportions for the inputs for \(A\) as:

\[ \frac{240}{1200} \text{ of } A, \quad \frac{720}{1200} \text{ of } B \text{ and } \frac{240}{1200} \text{ of primary.} \]

That for \(B\) are:

\[ \frac{750}{1500} \text{ of } A, \quad \frac{450}{1500} \text{ of } B \text{ and } \frac{300}{1500} \text{ of primary.} \]
21 Linear Inequalities

We know that $y = mx + b$ represents a straight line in the $xy$-plane. But what does the inequality $y \geq mx + b$ represent? Let us look at a concrete case: $y \geq 2x - 4$. Here the two variables $x$ and $y$ are related by an inequality, and the right side of the inequality is a linear function of $x$. Hence we call this a linear inequality.

The equation $y = 2x - 4$ has as its graph a straight line with slope 2 and $y$-intercept 4.

It turns out that the inequality $y \geq 2x - 4$ is satisfied by any point that lies on or above the straight line $y = 2x - 4$.

Therefore, all the points which satisfy $y \geq 2x - 4$ form a half plane that lies above the line $y = 2x - 4$. Similarly, the inequality $y \leq 2x - 4$ gives a half plane that lies below $y = 2x - 4$. If equality is not allowed, then $y > 2x - 4$ gives the half plane above $y = 2x - 4$ excluding the straight line $y = 2x - 4$. Similarly, $y < 2x - 4$ gives the half plane below $y = 2x - 4$ excluding this straight line itself.

**Example 1** Sketch the graph of the linear inequality $2x - 3y < 6$.

**Solution.** We know $2x - 3y = 6$ gives a straight line which can be rewritten as

$$-3y = 6 - 2x, \quad y = \frac{1}{-3}(6 - 2x) = -2 + \frac{2}{3}x$$

or

$$y = \frac{2}{3}x - 2.$$  

The inequality $2x - 3y < 6$ represents a half plane excluding the straight line $2x - 3y = 6$, but we need to determine whether the half plane is above or below the line $2x - 3y = 6$.

Adding $3y$ to both sides of the inequality we obtain

$$2x - 3y + 3y < 6 + 3y$$
\[2x < 6 + 3y\]

Subtracting 6 from both sides of the inequality we obtain
\[2x - 6 < 6 + 3y - 6\]
\[2x - 6 < 3y\]
or
\[3y > 2x - 6\]

Dividing both sides by 3, we get
\[y > \frac{2}{3}x - 2\]

Hence the half plane is above the straight line \[y = \frac{2}{3}x - 2\].

Note that if we subtract 2\(x\) from both sides of the original inequality, we deduce
\[-3y < 6 - 2x\]

and if we divide the inequality by \(-3\), we should reverse the inequality sign and obtain
\[y > \frac{6 - 2x}{-3} = -2 + \frac{2}{3}x\]

\[\square\]

**Example 2.** An investor plans to invest up to $30000 in two stocks, \(A\) and \(B\). Stock \(A\) is currently priced at $165 and stock \(B\) at $90 per share. If the investor buys \(x\) shares of \(A\) and \(y\) shares of \(B\), use an inequality to represent the relationship of \(x\) and \(y\).

**Solution.** The total cost of buying \(x\) shares of \(A\) and \(y\) shares of \(B\) is

\[165x + 90y\] dollars
As the investor plans to invest up to $30000 in buying these two shares, apparently we should have
\[165x + 90y \leq 30000\]

Very often, inequalities of more than two variables are needed. Then it is usually difficult to find a graph for the inequality or inequalities. One then has to rely on analytical tools to analyse them.

**Example 3.** An electronics company makes television sets at two factories, \(F_1\) and \(F_2\). \(F_1\) can produce up to 100 sets per week and \(F_2\) can produce up to 200 sets per week. The company has three distribution centres, \(X, Y\) and \(Z\). \(X\) requires 50 television sets per week, \(Y\) requires 75 sets per week, and \(Z\) requires 125 sets per week in order to meet the demands in their respective areas. If factor \(F_1\) supplies \(x\) sets per week to distribution centre \(X\), \(y\) sets to \(Y\), and \(z\) sets to \(Z\), write the inequalities satisfied by \(x, y,\) and \(z\).

**Solution.** It is better to use the following diagram to help us analyze the relationships between \(x, y,\) and \(z\).

Firstly we must have \(x \geq 0, y \geq 0, z \geq 0\).

Secondly, the total number of sets supplied by \(F_1\) should not be larger than 100. Hence
\[x + y + z \leq 100.\]

Thirdly, as \(X\) requires 50 sets per week, the supply by \(F_1\), should not be bigger than this number, i.e.
\[x \leq 50.\]

Similarly
\[y \leq 75, \quad z \leq 125.\]

As \(F_1\) supplies \(x\) sets to \(X\) and \(X\) requires 50 sets per week, \(F_2\) must supply \(50 - x\) sets to \(X\). Similarly, \(F_2\) must supply \(75 - y\) sets to \(Y\) and \(125 - z\) sets to \(Z\).
The total number of sets supplied by $F_2$ is then

$$(50 - x) + (75 - y) + (125 - z) = 250 - (x + y + z)$$

But this number should not be larger than 200 which $F_2$ can produce up to per week. Thus we have

$$250 - (x + y + z) \leq 200,$$

or

$$x + y + z \geq 50.$$ 

Thus we have eight inequalities:

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x \leq 50, \quad y \leq 75, \quad z \leq 125$$

$$x + y + z \leq 100, \quad x + y + z \geq 50$$
22 Linear Optimization

A linear programming problem is one that involves finding the maximum or minimum value of some linear algebraic expression when the variables in this expression are subject to a number of linear inequalities. The following example is typical of such problems.

Example 1 (Maximum Profit). A company manufactures two products, $X$ and $Y$. Each of these products requires a certain time on the assembly line and a further amount of time in the finishing shop. Each item of $X$ needs 5 hours for assembly and 2 hours for finishing, and each item of type $Y$ needs 3 hours for assembly and 4 hours for finishing. In any week, the firm has available 105 hours on the assembly line and 70 hours in the finishing shop. The firm can sell all it can produce and makes a profit of $200 on each item of $X$ and $160 on each item of $Y$. Find the number of items of each type that should be manufactured per week to maximize the total profit.

Solution It is usually convenient to summarize the given information in the form of a table. The following table shows the information given in Example 1.

<table>
<thead>
<tr>
<th></th>
<th>Assembly</th>
<th>Finishing</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>5</td>
<td>2</td>
<td>200</td>
</tr>
<tr>
<td>Y</td>
<td>3</td>
<td>4</td>
<td>160</td>
</tr>
<tr>
<td>Available</td>
<td>105</td>
<td>70</td>
<td></td>
</tr>
</tbody>
</table>

Suppose that the firm produces $x$ items of $X$ per week, and $y$ items of $Y$ per week. Then the time needed on the assembly line will be $5x$ hours for $X$ and $3y$ hours for $Y$, or $(5x + 3y)$ hours in all. Since only 105 hours are available, we must have

$$5x + 3y \leq 105$$

Similarly, considering the time needed in the finishing shop, we derive

$$2x + 4y \leq 70$$

Each item of $X$ produces a profit of 200 dollars, $x$ items produce $200x$ dollars in profit. Similarly, $y$ items of $Y$ produce $160y$ dollars in profit. Hence the total weekly profit $P$ (in dollars) is given by
\[ P = 200x + 160y \]

Therefore the problem is to find values of \( x \) and \( y \) that maximize

\[ P = 200x + 160y \]

when \( x \) and \( y \) are subject to the conditions

\[ 5x + 3y \leq 105, \quad 2x + 4y \leq 70, \quad x \geq 0, \quad y \geq 0 \]

Note that the inequalities \( x \geq 0, \ y \geq 0 \) are added for completeness. This follows from the definition of \( x \) and \( y \), and adding these two inequalities is necessary for the mathematical treatment of this problem.

To find \( x \) and \( y \) manually for this problem, one can use the methods in Chapter 11 of the text book. In this unit, we are satisfied with using Excel to find the solutions, and the detailed steps for using Excel to solve this kind of problems are given in the booklet “Workbook for Practical Classes”.

The solution to this problem (by Excel) is

\[ x = 15, \quad y = 10 \]

In a general linear programming problem, the inequalities that must be satisfied by the variables are called the **constraints**, and the linear function to be maximized or minimized is called the **objective function**. In Example 1, the constraints are:

\[ 5x + 3y \leq 105, \quad 2x + 4y \leq 70, \quad x \geq 0, \quad y \geq 0, \]

and the objective function is

\[ 200x + 160y \]

**Example 2** A chemical firm makes two brands of fertilizer. The regular brand contains nitrates, phosphates, and potash in the ratio 3:6:1 (by weight) and the super brand contains these three ingredients in the ratio 4:3:3. Each month the firm
can rely on a supply of 9 tons of nitrates, 13.5 tons of phosphates, and 6 tons of potash. The firm’s manufacturing plant can produce at most 25 tons of fertilizer per month. If the firm makes a profit of 300 dollars on each ton of regular fertilizer and 400 dollars on each ton of the super grade, what amounts of each grade should be produced in order to yield the maximum profit?

**Solution** The information given is summarized in the following table.

<table>
<thead>
<tr>
<th>Nitrates</th>
<th>Phosphates</th>
<th>Potash</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>0.3</td>
<td>0.6</td>
<td>0.1</td>
</tr>
<tr>
<td>Super</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Available</td>
<td>9</td>
<td>13.5</td>
<td>6</td>
</tr>
</tbody>
</table>

Let the firm produce \( x \) tons of regular grade and \( y \) tons of super grade. Then the constraints are

\[
x \geq 0, \quad y \geq 0, \quad 0.3x + 0.4y \leq 9, \quad 0.6x + 0.3y \leq 13.5 \quad 0.1x + 0.3y \leq 6
\]

and the objective function is

\[ P = 300x + 480y. \]

Using Excel, we obtain \( x = 6, y = 18 \).

**Example 3** A chemical company is designing a plant for producing two types of polymer, \( P_1 \) and \( P_2 \). The plant must be capable of producing at least 100 units of \( P_1 \) and 420 units of \( P_2 \) per day. There are two possible designs for the basic reaction chambers which are to be included in the plant: each chamber of type \( A \) costs 600,000 dollars and is capable of producing 10 units of \( P_1 \) and 20 units of \( P_2 \) per day; type \( B \) is a cheaper design costing 300,000 dollars and capable of producing 4 units of \( P_1 \) and 30 units of \( P_2 \) per day. For some reason it is necessary to have at least 4 chambers of each type in the plant. How many chambers of each type should be included to minimize the cost while still meet the required production schedule?

**Solution** The given information is summarized in the following table.

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>Cost (Thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chamber A</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Chamber B</td>
<td>4</td>
<td>30</td>
</tr>
<tr>
<td>Required</td>
<td>100</td>
<td>420</td>
</tr>
</tbody>
</table>

Let the design include \( x \) chambers of type \( A \) and \( y \) chambers of type \( B \). Then the constraints are
\[ x \geq 4, \quad y \geq 4, \quad 10x + 4y \geq 100, \quad 20x + 30y \geq 420 \]

We want to minimize the objective function

\[ C = 600x + 300y \]

Using Excel, it gives \( x = 6 \) and \( y = 10 \). \( \square \)
23 The Derivative

Let \( y = f(x) \) be a function. When the value of the variable \( x \) changes from \( x_0 \) to \( x_0 + \Delta x \), where \( \Delta x \) is called the **increment** in \( x \), then the value of \( y \) changes from \( f(x_0) \) to \( f(x_0 + \Delta x) \) and the increment in \( y \) is \( \Delta y = f(x_0 + \Delta x) - f(x_0) \). Very often, we need to know the rate of change \( \frac{\Delta y}{\Delta x} \), which is called the **average rate of change** of the function \( f \) over the range of \( x \) between \( x_0 \) and \( x_0 + \Delta x \).

For example, the distance \( S \) traveled by an object is a function of time \( t \), \( S = S(t) \), and the average speed of the object between time \( t_0 \) and \( t_0 + \Delta t \) is the average rate of change of the function \( S \), \( \frac{\Delta S}{\Delta t} = \frac{S(t_0 + \Delta t) - S(t_0)}{\Delta t} \).

**Example 1** A chemical manufacturer finds that the cost per week of making \( x \) tons of a certain fertilizer is given by

\[
C(x) = 20000 + 40x^2 \text{ (dollars)},
\]

Find the average rate of change of cost per extra ton produced from a certain \( x_0 \) tons per week.

**Solution** When \( x \) changes from \( x_0 \) to \( x_0 + 1 \), \( C(x) \) changes from \( C(t_0) \) to \( C(x_0 + 1) \), and

\[
\Delta C = C(t_0 + 1) - C(x_0) = 20000 + 40(x_0 + 1)^2 - (20000 + 40x_0^2) \\
= 40(x_0 + 1)^2 - 40x_0^2 = 40(x_0^2 + 2x_0 + 1) - 40x_0^2 = 80x_0 + 40
\]

Thus

\[
\frac{\Delta C}{\Delta x} = \frac{80x_0 + 40}{1} = 80x_0 + 40.
\]

Hence, the average rate of change is 80\(x_0 \) + 40 dollars per extra ton produced from \( x_0 \).

In many instances in both science and everyday life, the average rate of change does not provide the information of most importance. For example, if a person traveling in an automobile hits a concrete wall, it is not the average speed but the speed at the instant of collision that determines whether the person will survive the accident.

Suppose the distance \( S \) (in meters) traveled by an object at time \( t \) (in seconds) is given by

\[
S = S(t) = 16t^2
\]
To find the speed at \( t = 3 \), we look at the average speed between 3 and \( 3 + \Delta t \), which is

\[
\frac{\Delta S}{\Delta t} = \frac{S(3 + \Delta t) - S(3)}{\Delta t} = \frac{16(3 + \Delta t)^2 - 16(3)^2}{\Delta t} = \frac{16(9 + 6\Delta t + \Delta t^2) - 16 \times 9}{\Delta t} = \frac{16 \times 6\Delta t + 16\Delta t^2}{\Delta t} = 96 + 16\Delta t
\]

The following table gives the values for \( \frac{\Delta S}{\Delta t} \) for a series of values of small \( \Delta t \).

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>0.5</th>
<th>0.25</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\Delta S}{\Delta t} )</td>
<td>104</td>
<td>100</td>
<td>97.6</td>
<td>96.16</td>
<td>96.016</td>
</tr>
</tbody>
</table>

It is clear that as \( \Delta t \) gets smaller and smaller, \( \frac{\Delta S}{\Delta t} \) gets closer and closer to 96, which is also clear from the expression

\[
\frac{\Delta S}{\Delta t} = 96 + 16\Delta t
\]

We say that the limit of \( \frac{\Delta S}{\Delta t} \) is 96 as \( \Delta t \) approaches 0, and 96 is the speed of the object at \( t = 3 \). We write this as

\[
\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = 96
\]

In general, let \( y = f(x) \) be a given function. If the average rate of change \( \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \) gets closer and closer to a certain number \( l \) when \( \Delta x \) gets closer and closer to 0, that is, if \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = l \), then we say the function \( y = f(x) \) is differential at \( x = x_0 \), and call \( l \) the derivative of \( f(x) \) at \( x = x_0 \), and denote it by \( \frac{dy}{dx} \bigg|_{x_0} \) or \( f'(x_0) \). If \( f \) is differential at a general value \( x \), we usually write just \( \frac{dy}{dx} \) or \( f'(x) \) for the derivative, which is again a function of \( x \).

**Example 2** Find \( f'(x) \) if \( f(x) = x^2 + 2x \). Evaluate \( f'(2) \) and \( f'(-1) \).

**Solution**
\[
\Delta y = f(x + \Delta x) - f(x) = [(x + \Delta x)^2 + 2(x + \Delta x)] - (x^2 + 2x) \\
= x^2 + 2\Delta x(x) + (\Delta x)^2 + 2x + 2\Delta x - x^2 - 2x \\
= 2(\Delta x)x + (\Delta x)^2 + 2\Delta x \\
\frac{\Delta y}{\Delta x} = \frac{2(\Delta x)x + (\Delta x)^2 + 2\Delta x}{\Delta x} = 2x + \Delta x + 2 \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(2x + \Delta x + 2)}{\Delta x} = 2x + 2 \\
\]

Therefore, \( f'(x) = 2x + 2 \), and

\[
\begin{align*}
  f'(2) &= 2(2) + 2 = 6 \\
  f'(-1) &= 2(-1) + 2 = 0 \\
\end{align*}
\]

\( \square \)

The derivative \( f'(x) \) has a geometric interpretation. We know that \( y = f(x) \) represents a curve in the \( xy \)-plane. Let \( P \) and \( Q \) denote the points \((x, f(x))\) and \((x + \Delta x, f(x + \Delta x))\), respectively. Then the slope of the line segment \( PQ \) is

\[
\frac{(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}
\]

As \( \Delta x \) becomes smaller and smaller, the point \( Q \) moves closer and closer to \( P \) and the chord segment \( PQ \) becomes more and more nearly a tangent. As \( \Delta x \) approaches \( O \), the slope of the chord \( PQ \) approaches the slope of the tangent line at \( P \). But

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)
\]

Hence \( f'(x) \) is the slope of the tangent line to the curve \( y = f(x) \) at the point \((x, f(x))\).
24 Derivatives of Power Functions

It is clear from the last lecture that finding derivatives of functions by direct use of the definition of the derivative is not always easy and is generally time consuming. This task can be appreciably lightened by the use of certain standard formulas. Here we concentrate on such formulas for power functions.

Recall that a power function has the form \( y = x^n, \) where \( n \) is a constant.

**Theorem 1**

If \( y = x^n, \) then \( \frac{dy}{dx} = nx^{n-1}, \) or, equivalently \( (x^n)' = nx^{n-1} \)

From this general formula, we can get

(a) \( \frac{d}{dx}(x^6) = 6x^{6-1} = 6x^5 \)

(b) \( \frac{d}{dt}(\sqrt{t}) = \frac{d}{dt}(t^{\frac{1}{2}}) = \frac{1}{2}t^{\frac{1}{2}-1} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}} \)

(c) \( \frac{d}{dy}(\frac{1}{y}) = \frac{d}{dy}(y^{-1}) = (-1)y^{-1-1} = (-1)y^{-2} = -\frac{1}{y^2} \)

(d) \( \frac{d}{du}(u) = \frac{d}{du}(u^1) = 1 \cdot u^{1-1} = 1 \cdot u^0 = 1 \)

(e) \( \frac{d}{dx}(1) = \frac{d}{dx}(x^0) = 0 \cdot x^{0-1} = 0 \cdot x^{-1} = 0 \)

Theorem 1 is proved by using the definition of derivative, and the proof is not trivial. The interested reader is referred to any calculus book for a proof, but this is not required for this unit.

The formula in Theorem 1 itself is not very useful in calculating derivatives of functions. However, it becomes very powerful when used together with the basic properties of the derivatives, some of which are stated in the following theorem.

**Theorem 2**

(a) If \( u(x) \) is a differentiable function of \( x \) and \( c \) is constant, then

\[
\frac{d}{dx}(cu) = c\frac{du}{dx}, \quad \text{or} \quad (cu)' = cu'
\]
(b) If \( u(x) \) and \( v(x) \) are two differentiable functions of \( x \), then

\[
\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}, \quad \text{or} \quad (u + v)' = u' + v'
\]

\[
\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}, \quad \text{or} \quad (u - v)' = u' - v'
\]

Again we skip the proof of this theorem, but concentrate on its applications.

**Example 1** Find \( \frac{dy}{dx} \) if \( y = x^3 + \sqrt{x} \).

**Solution**

\[
\frac{dy}{dx} = \frac{d}{dx}(x^3 + \sqrt{x}) = \frac{d}{dx}(x^3) + \frac{d}{dx}(\sqrt{x}) = 3x^2 + \frac{1}{2}x^{\frac{1}{2}-1} = 3x^2 + \frac{1}{2}\sqrt{x}
\]

\[\square\]

**Example 2** Find \( f'(x) \) where \( f(x) = 2x^2 + 3x + 6 - x^4 \)

**Solution**

\[
f'(x) = (2x^2)' + (3x)' + (6)' - (x^4)'
\]

\[
= 2(2x^2) + 3(1) + 6(0) - 4x^3
\]

\[
= 4x + 3 - 4x^3
\]

\[\square\]

**Example 3** Find \( \frac{du}{dt} \) where \( u = \frac{5t^4 + 7t - 3}{2t^3} \)

**Solution** We first simplify the expression for \( u \).

\[
u = \frac{5t^4 + 7t - 3}{2t^3} = \frac{5t^4}{2t^3} + \frac{7t}{2t^3} - \frac{3}{2t^3}
\]

\[
= \frac{5}{2}t + \frac{7}{2}t^{-2} - \frac{3}{2}t^{-3}
\]

Hence

\[
\frac{du}{dt} = \frac{d}{dt}\left(\frac{5}{2}t + \frac{7}{2}t^{-2} - \frac{3}{2}t^{-3}\right)
\]

\[
= \frac{5}{2} \frac{d}{dt}(t) + \frac{7}{2} \frac{d}{dt}(t^{-2}) - \frac{3}{2} \frac{d}{dt}(t^{-3})
\]

\[
= \frac{5}{2} \cdot 1t^{1-1} + \frac{7}{2}(-2)t^{-2-1} - \frac{3}{2}(-3)t^{-3-1}
\]
Example 4  Find the derivative of the function 
\[(\sqrt{x} + 1)^2 + (2x + 1)(x - 1)\]

Solution  We first simplify the expression of the function,

\[
y = (\sqrt{x} + 1)^2 + (2x + 1)(x - 1)
\]
\[
= (\sqrt{x})^2 + 2\sqrt{x} + 1 + 2x^2 - 2x + x - 1
\]
\[
= x + 2\sqrt{x} + 1 + 2x^2 - x - 1
\]
\[
= 2\sqrt{x} + 2x^2
\]

Hence  
\[
\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x}) + \frac{d}{dx}(2x^2)
\]
\[
= 2 \cdot \frac{1}{2}x^{1-1} + 2 \cdot 2x^{2-1}
\]
\[
= x^{-\frac{1}{2}} + 4x
\]
\[
= \frac{1}{\sqrt{x}} + 4x
\]
Marginal Analysis

Derivatives have a number of applications in business and economics in constructing what are called marginal rates. In this field, the word “marginal” is used to indicate a derivative, that is, a rate of change. For example, if \( C(x) \) is the cost function, where \( x \) represents the number of units, then \( C'(x) \) is called the marginal cost when \( x \) units are produced. Similarly, if \( R(x) \) is the revenue function, then \( R'(x) \) is called the marginal revenue, etc.

Example 1 (Marginal Cost) The cost function

\[
C(x) = 0.001x^3 - 0.3x^2 + 40x + 1000
\]

determines the cost as a function of \( x \). Evaluate the marginal cost when the production is given by \( x = 50 \), \( x = 100 \), and \( x = 150 \).

Solution The marginal cost is

\[
C'(x) = (0.001x^3 - 0.3x^2 + 40x + 1000)' = 0.003x^2 - 0.6x + 40
\]

When \( x = 50 \)

\[
C'(50) = 0.003(50)^2 - 0.6(50) + 40 = 17.5
\]

When \( x = 100 \)

\[
C'(100) = 0.003(100)^2 - 0.6(100) + 40 = 10
\]

When \( x = 150 \)

\[
C'(150) = 0.003(150)^2 - 0.6(150) + 40 = 17.5
\]

Roughly speaking, we can say that the 51st item costs $17.50 to produce, the 101st item costs $10, and the 151st item costs $17.50. (Such statements as these are not quite accurate, since the derivative gives the rate for an infinitesimally small increment in production, not for a unit increment). 

It is important not to confuse the marginal cost with the average cost. If \( C(x) \) is the cost function, then the average cost of producing \( x \) items is given by
Average Cost per item = \( \frac{C(x)}{x} \)

This is commonly denoted by \( \overline{C}(x) \), i.e.,

\[
\overline{C}(x) = \frac{C(x)}{x}
\]

**Example 2** For the cost function \( C(x) = 1000 + 10x + 0.1x^2 \), the marginal cost is \( C'(x) = 10 + 0.2x \). The average cost of producing \( x \) items is

\[
\overline{C}(x) = \frac{C(x)}{x} = \frac{1000 + 10x + 0.1x^2}{x} = 1000x^{-1} + 10 + 0.1x
\]

\( \square \)

**Example 3** (Marginal Revenue) Determine the marginal revenue when \( x = 300 \) if the demand equation is

\[ x = 1000 - 100p \]

**Solution** We know that revenue \( R = xp \). From the demand equation, we can solve for \( p \) in terms of \( x \):

\[
\begin{align*}
x &= 1000 - 100p, \\
100p &= 1000 - x \\
p &= \frac{1000 - x}{100} = 10 - 0.01x
\end{align*}
\]

Thus

\[
R = xp = x(10 - 0.01x) = 10x - 0.01x^2.
\]

The marginal revenue is

\[
R'(x) = (10x - 0.01x^2)' = 10 - 0.02x
\]

When \( x = 300 \), the marginal revenue is

\[
R'(300) = 10 - 0.02(300) = 10 - 6 = 4
\]
Example 4 (Marginal Profit) The demand equation for a certain item is

\[ p + 0.1x = 80 \]

and the cost function is

\[ C(x) = 5000 + 20x. \]

Compute the marginal profit when (a) 150 units are produced and sold, (b) when 400 units are produced and sold. Find \( x \) that maximizes the profit and find the marginal profit at this value of \( x \).

Solution The profit \( P(x) = R(x) - C(x) \), where \( R(x) \) is the revenue function.

We have \( R = xp \), and by the demand equation,

\[ p + 0.1x = 80, \quad p = 80 - 0.1x \]

Therefore,

\[ R(x) = xp = x(80 - 0.1x) = 80x - 0.1x^2. \]

It follows that

\[
\begin{align*}
P(x) &= R(x) - C(x) \\
     &= (80x - 0.1x^2) - (5000 + 20x) \\
     &= 80x - 0.1x^2 - 5000 - 20x \\
     &= 60x - 0.1x^2 - 5000
\end{align*}
\]

\[ P'(x) = (60x - 0.1x^2 - 5000)' = 60 - 0.2x \]

When \( x = 150 \),

\[ P'(150) = 60 - 0.2(150) = 60 - 30 = 30 \]

Thus when 150 items are being produced and sold, the marginal profit, that is, roughly speaking the extra profit per additional item is $30.

When \( x = 400 \),

\[ P'(400) = 60 - 0.2(400) = 60 - 80 = -20 \]

Thus when 400 items are being produced and sold, the extra profit per additional item is -$20, or a loss of $20 per additional item.
Since the profit \( P(x) = 60x - 0.1x^2 - 5000 \) is a quadratic function of the form \( ax^2 + bx + c \) with \( a = -0.1, b = 60 \) and \( c = -5000 \), we know that it takes a maximum when

\[
x = \frac{-b}{2a} = \frac{-60}{2(-0.1)} = \frac{60}{0.2} = 300
\]

and

\[
P(300) = 60(300) - 0.1(300)^2 - 5000 = 18000 - 9000 - 5000 = 4000
\]

\[
P'(300) = 60 - 0.2(300) = 60 - 60 = 0
\]

It is interesting to see from the above example that at \( x = 300 \), where maximal profit is achieved, the marginal profit is 0.
26 Derivatives of Products and Quotients

In this lecture, we are to learn two more important properties of derivatives, namely, the **product rule** and the **quotient rule**.

**Theorem 1** (Product Rule). If \( u(x) \) and \( v(x) \) are two differentiable functions, then

\[
\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}, \quad \text{i.e., } (uv)' = uv' + u'v
\]

A proof of the theorem can be found in the textbook, but that is not required in this unit. Instead, we focus on the applications of this theorem.

**Example** Find \( y' \) if \( y = (x^2 + 1)(3x^2 + 6x + 2) \).

**Solution:** The given function \( y \) can be regarded as the product of the two functions \( u = x^2 + 1 \) and \( v = 3x^2 + 6x + 2 \). By the product rule,

\[
y' = (uv)' = uv' + u'v
\]

But we have

\[
u' = (3x^2 + 6x + 2)' = 6x + 6,
\]

\[
u' = (x^2 + 1)' = 2x.
\]

Hence

\[
y' = uv' + u'v = (x^2 + 1)(6x + 6) + 2x(3x^2 + 6x + 2)
\]

\[
= 6x^3 + 6x + 6x^2 + 6 + 6x^3 + 12x^2 + 4x
\]

\[
= 12x^3 + 18x^2 + 10x + 6.
\]

\[\square\]

**Remark.** In example 1, one could also simplify the expression of the function first and then calculate the derivative, as in the following.

\[
y = (x^2 + 1)(3x^2 + 6x + 2) = 3x^4 + 6x^3 + 2x^2 + 3x^2 + 6x + 2
\]

\[
= 3x^4 + 6x^3 + 5x^2 + 6x + 2.
\]

\[
y' = (3x^4 + 6x^3 + 5x^2 + 6x + 2)' = 12x^3 + 18x^2 + 10x + 6
\]

**Example 2** Find the marginal revenue if the demand equation is given by \( p = 30 - 0.1x^2 \).
Solution. The revenue is given by
\[ R(x) = xp \]
where \( p \) is a function of \( x \) determined by the demand equation.

We can use the product rule to obtain
\[
R'(x) = (xp)' = xp' + (x)'p = x(30 - 0.1x^2)' + 1 \cdot (30 - 0.1x^2) = x(-0.2x) + 30 - 0.1x^2 = -0.3x^2 + 30
\]

\[ \square \]

**Theorem 2** (Quotient Rule) If \( u(x) \) and \( v(x) \) are differentiable functions, then
\[
\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}, \text{ i.e., } \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}
\]

Again we refer to the text book for the proof of this theorem while concentrating on its applications in this unit.

**Example 3.** Find \( y' \) if \( y = \frac{x^2 + 1}{x^3 + 2} \).

**Solution.** We can regard the function \( y \) as the quotient \( \frac{u}{v} \) where \( u = x^2 + 1 \) and \( v = x^3 + 2 \). By the quotient rule,
\[
y' = \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}.
\]

But \( u' = (x^2 + 1)' = 2x, \ v' = (x^3 + 2)' = 3x^2 \). Therefore we have
\[
y' = \frac{u'v - uv'}{v^2} = \frac{2x(x^3 + 2) - (x^2 + 1) \cdot 3x^2}{(x^3 + 2)^2} = \frac{2x^4 + 4x - 3x^4 - 3x^2}{(x^3 + 2)^2} = \frac{-x^4 - 3x^2 + 4x}{(x^3 + 2)^2}
\]

\[ \square \]

**Example 4.** The gross national product of a certain country is increasing with time according to the formula \( I = 100 + t \) (billions of dollars), where \( t \) is measured
in years from 1980. If the population is determined by the formula \( P = 75 + 2t \) (millions), find the rate of change of per capita income in year 2000.

**Solution.** The per capita income, which we denote by \( y \), is equal to the \( GNP \) divided by the population size:

\[
y = \frac{I}{P} = \frac{(100 + t)10^9}{(75 + 2t)10^6} = \frac{(100 + t)}{75 + 2t} \cdot 10^3 \text{ (dollars)}
\]

\[
= \frac{100 + t}{75 + 2t} \text{ (thousands of dollars)}.
\]

To find the rate of change of \( y \), we calculate its derivative

\[
y' = \left( \frac{100 + t}{75 + 2t} \right)' = \frac{u'v - uv'}{v^2},
\]

where \( u = 100 + t, v = 75 + 2t \). We have

\[
u' = (100 + t)' = 1, \quad v' = (75 + 2t)' = 2.
\]

Therefore,

\[
y' = \frac{u'v - uv'}{v^2} = \frac{(1)(75 + 2t) - (100 + t)(2)}{(75 + 2t)^2} = \frac{75 + 2t - 200 - 2t}{(75 + 2t)^2} = \frac{-125}{(75 + 2t)^2}
\]

In 2000, \( t = 20 \) and

\[
y' = \frac{-125}{(75 + 2 \times 20)^2} = \frac{-125}{(115)^2} = -0.0095
\]

\[ \square \]

**Example 5.** Find \( \frac{dy}{dx} \) if \( y = \frac{(x + 1)(x^3 - 2)}{x^2 + 1} \)

**Solution.** Let \( u = (x + 1)(x^3 - 2), \quad v = x^2 + 1 \). Then

\[
\frac{dy}{dx} = \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2} \quad \text{(quotient rule)}
\]
We have

\[
\begin{align*}
u' &= [(x + 1)(x^3 - 2)]' \\
&= (x + 1)(x^3 - 2)' + (x + 1)'(x^3 - 2) \quad \text{(product rule)} \\
&= (x + 1)(3x^2) + (1)(x^3 - 2) \\
&= 3x^3 + 3x^2 + x^3 - 2 \\
&= 4x^3 + 3x^2 - 2.
\end{align*}
\]

\[v' = (x^2 + 1)' = 2x.\]

So

\[
\begin{align*}
\frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(4x^3 + 3x^2 - 2)(x^2 + 1) - (x + 1)(x^3 - 2)(2x)}{(x^2 + 1)^2} \\
&= \frac{4x^5 + 3x^4 - 2x^2 + 4x^3 + 3x^2 - 2 - (x^4 + x^3 - 2x - 2)(2x)}{(x^2 + 1)^2} \\
&= \frac{4x^5 + 3x^4 + 4x^3 + x^2 - 2 - 2x^5 - 2x^4 + 4x^2 + 4x}{(x^2 + 1)^2} \\
&= \frac{2x^5 + x^4 + 4x^3 + 5x^2 + 4x - 2}{(x^2 + 1)^2}
\end{align*}
\]
27 The Chain Rule

Let \( y = f(u) \) be a function of \( u \) and \( u = g(x) \) be a function of \( x \). Then the composite function

\[
y = f[g(x)]
\]

is a function of \( x \). The derivative of composite functions can be found by the use of the following theorem, known as the **Chain Rule**. Its proof can be found in the text book.

**Theorem** (Chain Rule). If \( y \) is a function of \( u \) and \( u \) is a function of \( x \), then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

The chain rule provides what is probably the most useful of all the aids to differentiation, as will soon become clear. It is a tool that is constantly in use when you work with the differential calculus, and you should master its use as soon as possible. When using it to differentiate a complicated function, it is necessary at the start to spot how to write the given function as the composition of two simpler functions. The following examples provide some illustrations.

**Example 1.** Find \( \frac{dy}{dx} \) when \( y = (x^2 + 1)^6 \).

**Solution.** We could solve this problem by expanding \( (x^2 + 1)^6 \) as a polynomial in \( x \). However, it is much simpler to use the chain rule.

A key step is the observation that \( y \) can be written as a composite function in the following way.

\[
y = u^6, \quad u = x^2 + 1.
\]

Then

\[
\frac{dy}{du} = (u^6)' = 6u^5, \quad \frac{du}{dx} = (x^2 + 1)' = 2x.
\]

From the chain rule, we have

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u^5 \cdot 2x = 6(x^2 + 1)^5 \cdot 2x = 12x(x^2 + 1)^5.
\]
Another way of writing the chain rule is that if \( y = f(u) \), then

\[
\frac{dy}{dx} = f'(u) \frac{du}{dx}
\]

Think of a composite function as having different layers that you peel off one by one. The outside layer of the function corresponds to the part you would compute last if you were evaluating it. For example, if \( y = (x^2 + 1)^6 \), the outside part of the function is the sixth power and the inside part is \((x^2 + 1)\). If you were evaluating \( y \) for a particular value of \( x \), you would first evaluate the inside part, \( x^2 + 1 \), and then you would raise it to the sixth power. For example, if \( x = 1 \), then inside \( = x^2 + 1 = 1^2 + 1 = 2 \) and \( y = (\text{inside})^6 = 2^6 = 64 \). Then

\[
\frac{dy}{dx} = 6(\text{inside})^5 \cdot \frac{d}{dx} (\text{inside})
= 6(x^2 + 1)^5 \cdot \frac{d}{dx} (x^2 + 1)
= 6(x^2 + 1)^5 \cdot 2x
= 12x(x^2 + 1)^5.
\]

**Example 2.** Given \( f(t) = \frac{1}{\sqrt{t^2 + 3}} \), find \( f'(t) \).

**Solution.** Let \( u = t^2 + 3 \), so that \( y = f(t) = \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}} \).

Then

\[
\frac{dy}{du} = \left( u^{-\frac{1}{2}} \right)' = -\frac{1}{2} u^{-\frac{3}{2}} = -\frac{1}{2} u^{-\frac{3}{2}},
\]

\[
\frac{du}{dt} = (t^2 + 3)' = 2t
\]

Thus, by the chain rule,

\[
\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = -\frac{1}{2} u^{-\frac{3}{2}} \cdot 2t = -\frac{1}{2} (t^2 + 3)^{-\frac{3}{2}} \cdot 2t = -t (t^2 + 3)^{-\frac{3}{2}}.
\]

**Remark.** In example 2, we can also solve directly as follows.

\[
f(t) = \frac{1}{\sqrt{t^2 + 3}} = (t^2 + 3)^{-\frac{1}{2}}
\]
The inside is \((t^2 + 3)\) and outside is the power \(-\frac{1}{2}\). Using the formula for power function to differentiate the outside part, we have
\[
f'(t) = -\frac{1}{2} (t^2 + 3)^{-\frac{3}{2}} \cdot \frac{d}{dt} (t^2 + 3)
\]
\[
= -\frac{1}{2} (t^2 + 3)^{-\frac{3}{2}} \cdot 2t = -t (t^2 + 3)^{-\frac{3}{2}}.
\]

**Example 3** Given \(y = (x^2 + 5x + 1)(2 - x^2)^4\), find \(\frac{dy}{dx}\).

**Solution.** First we write \(y\) as a product, \(y = uv\), where \(u = x^2 + 5x + 1\) and \(v = (2 - x^2)^4\). By the product rule, we have
\[
\frac{dy}{dx} = u'v + uv'
\]

We have \(u' = (x^2 + 5x + 1)' = 2x + 5\), but to find \(v'\), we must use the chain rule. For this, inside = \((2 - x^2)\) and the outside part of \(v\) is the fourth power. Thus
\[
v' = 4(2 - x^2)^3 \cdot \frac{d}{dx}(2 - x^2)
\]
\[
= 4(2 - x^2)^3 \cdot (-2x)
\]
\[
= -8x(2 - x^2)^3.
\]

Finally
\[
\frac{dy}{dx} = u'v + uv'
\]
\[
= (x^2 + 5x + 1) \cdot -8x(2 - x^2)^3 + (2x + 5) \cdot (2 - x^2)^4
\]
\[
= -8x(x^2 + 5x + 1)(2 - x^2)^3 + (2x + 5)(2 - x^2)^4.
\]

**Example 4.** Find \(\frac{dw}{dx}\) if \(w = \left(\frac{x-1}{x+1}\right)^3\).

**Solution.** Here we have a choice as to how we break this function down. We can write \(w\) as a composite function,
\[
w = u^3, \quad u = \frac{x-1}{x+1}
\]
and then use the chain rule. Alternatively, we can write \(y = \frac{u}{v}\) where \(u = (x-1)^3\) and \(v = (x+1)^3\) and then use the quotient rule. Or a third alternative is to write
\[ y = uv \text{ where } u = (x - 1)^3 \text{ and } v = (x + 1)^{-3} \text{ and use the product rule. We shall use the first of these methods, but you might like to check that the other methods give the same answer.} \]

By the chain rule, we obtain from the expressions in (1),

\[
\frac{dv}{dx} = \frac{dw}{du} \cdot \frac{du}{dx}
\]

\[= (u^3)' \cdot \frac{du}{dx} = 3u^2 \frac{du}{dx} = 3 \left( \frac{x - 1}{x + 1} \right)^2 \frac{du}{dx}.\]

To find \( \frac{du}{dx} \) we write \( u = \frac{u_1}{v_1} \), where \( u_1 = x - 1 \) and \( v_1 = x + 1 \) and use the quotient rule,

\[
\frac{du}{dx} = \frac{u_1'v_1 - u_1v_1'}{v_1^2} = \frac{(1)(x + 1) - (x - 1)(1)}{(x + 1)^2} = \frac{2}{(x + 1)^2}.
\]

Thus, finally,

\[
\frac{dw}{dx} = 3 \left( \frac{x - 1}{x + 1} \right)^2 \cdot \frac{du}{dx} = 3 \left( \frac{x - 1}{x + 1} \right)^2 \cdot \frac{2}{(x + 1)^2} = \frac{6(x - 1)^2}{(x + 1)^4}.
\]
28 Derivatives of Exponential and Logarithmic Functions

Consider the exponential function \( f(x) = a^x \). By definition,

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.
\]

But we have

\[
f(x + \Delta x) - f(x) = a^{x+\Delta x} - a^x = a^x \cdot a^{\Delta x} - a^x = a^x(a^{\Delta x} - 1)
\]

Hence

\[
\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{a^x(a^{\Delta x} - 1)}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x}.
\]

If \( \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = l \), then we must have

\[
f'(x) = \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x} = a^x \cdot l.
\]

It can be shown (but this is beyond the scope of this unit) that the limit \( l \) always exists and its value depends on \( a \). Moreover, \( l = 1 \) if \( a = e = 2.71828 \ldots \), the base of the natural logarithmic function. Thus we have the following formula.

\[
\text{If } y = e^x, \text{ then } \frac{dy}{dx} = e^x, \text{ i.e. } (e^x)' = e^x
\]

The reasons that the natural exponential function is so important rests in this property that its derivative is everywhere equal to the function itself. It is, apart from a constant factor, the only function that possesses this property. It is this fact that accounts for our interest in the number \( e \) and in exponential expressions and logarithm that have \( e \) as their basis.

Example 1. Evaluate \( f'(x) \) if \( f(x) = x^2 e^x \).

Solution. We use the product rule, regarding \( f(x) = uv \) with \( u = x^2 \) and \( v = e^x \).

\[
f'(x) = (uv)' = uv' + u'v
\]

\[
= (x^2)(e^x)' + (x^2)'(e^x)
\]

\[
= x^2e^x + 2xe^x
\]

\[
= (x^2 + 2x)e^x
\]
Example 2. Find $\frac{dy}{dx}$ if $y = e^{x^3}$

Solution. Here we break $y$ up as a composite function, $y = e^u, u = x^3$. Then, by the chain rule

$$
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (e^u)'(x^3)' = e^u \cdot (3x^2) = 3x^2e^{x^3}
$$

In general, we have, by the chain rule

$$
(e^{u(x)})' = u'(x)e^{u(x)}
$$

Example 3. (Population Growth). A population grows according to the logistic model such that at time $t$ its size $y$ is given by

$$
y = y_m(1 + Ce^{-kt})^{-1}
$$

with $y_m, C$ and $k$ constants. Find the rate of increase of population at time $t$.

Solution. The required rate of increase is $\frac{dy}{dt}$. We observe that $y$ is a composite function of $t$ of the form

$$
y = y_m u^{-1}, \quad u = 1 + Ce^{-kt}.
$$

Therefore,

$$
\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}
$$

We have

$$
\frac{dy}{du} = (y_m u^{-1})' = y_m (-1) u^{-2} = -y_m u^{-2}
$$

$$
\frac{du}{dt} = (1 + Ce^{-kt})' = C(e^{-kt})' = C(-k)e^{-kt} = -kCe^{-kt}
$$

where we used the chain rule to obtain $(e^{-kt})' = -ke^{-kt}$. 
Therefore,
\[
\frac{dy}{dt} = (-ymu^{-2})(-kCe^{-kt}) = ymkCu^{-2}e^{-kt}
\]
\[
= ymkC(1 + Ce^{-kt})^{-2}e^{-kt}.
\]

Let us now evaluate the derivative of the natural logarithm function \( y = \ln x \). If \( y = \ln x \), then \( x = e^y \), i.e.
\[
x = e^{\ln x}
\]
Taking the derivative to the functions of both sides of the above identity, we obtain
\[
(x)' = (e^{\ln x})'
\]
\[
1 = (e^{\ln x})'
\]
By the chain rule,
\[
(e^{\ln x})' = (\ln x)'e^{\ln x} = (\ln x)'x
\]
Therefore,
\[
1 = (\ln x)'x
\]
and
\[
(\ln x)' = \frac{1}{x}
\]
Thus, we have the formula,

\[
\text{If } y = \ln x, \text{ then } \frac{dy}{dx} = \frac{1}{x}, \text{ i.e. } (\ln x)' = \frac{1}{x}
\]

**Example 4** Find \( \frac{dy}{dx} \) if \( y = \ln(x^2 + x) \).

**Solution** We break \( y \) up as a composite function,
\[
y = \ln u, \ u = x^2 + x.
\]
By the chain rule,
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\ln u)'(x^2 + x)'
\]
\[
= \frac{1}{u}(2x + 1) = \frac{2x + 1}{x^2 + x}.
\]
Example 5. Find $\frac{dy}{dx}$ when $y = a^x$

Solution Using the fact that $\alpha = e^{\ln \alpha}$ for any positive number $\alpha$, we can write

$$a^x = e^{\ln(a^x)} = e^{x \ln a}$$

By the chain rule,

$$(a^x)' = (e^{x \ln a})' = (x \ln a)' e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x.$$

Similarly, we can use

$$\log_a x = \frac{\ln x}{\ln a}$$

to deduce

$$(\log_a x)' = \frac{1}{(\ln a)x}.$$

Example 6 Find $\frac{dy}{dx}$ when $y = \ln \left(\frac{e^x}{\sqrt{x+1}}\right)$.

Solution We can start with the chain rule, with

$$y = \ln u, u = \frac{e^x}{\sqrt{x+1}}.$$ 

But it is much simpler if we start with simplifying the given function,

$$y = \ln \left(\frac{e^x}{\sqrt{x+1}}\right) = \ln(e^x) - \ln(\sqrt{x+1}) = x - \ln(\sqrt{x+1}^2) = x - \frac{1}{2} \ln(x + 1).$$

Now

$$\frac{dy}{dx} = (x - \frac{1}{2} \ln(x + 1))' = (x)' - \frac{1}{2}(\ln(x + 1))' = 1 - \frac{1}{2} \frac{1}{x+1} = 1 - \frac{1}{2(x+1)}.$$
Now that we have introduced the derivatives of the exponential and logarithmic functions, let us summarize the three special forms of the chain rule that we shall use most.

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( [u(x)]^n )</th>
<th>( e^{u(x)} )</th>
<th>( \ln u(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>( n[u(x)]^{n-1}u'(x) )</td>
<td>( e^{u(x)}u'(x) )</td>
<td>( \frac{u'(x)}{u(x)} )</td>
<td></td>
</tr>
</tbody>
</table>
29 Derivatives and Graphs of Functions

In this lecture, we consider the significance of the derivative of a function as it relates to the graph. We will see how the derivative can be used to analyze the change of the values of the function as the independent variable changes.

**Definition** A function \( y = f(x) \) is said to be an **increasing function** over an interval of values of \( x \) if \( y \) increases with increase of \( x \). That is, if \( x_1 \) and \( x_2 \) are any two values in the given interval with \( x_2 > x_1 \), then \( f(x_2) > f(x_1) \).

A function \( y = f(x) \) is said to be a **decreasing function** over an interval of its domain if \( y \) decreases with increase of \( x \). That is, if \( x_2 > x_1 \) are two values of \( x \) in the given interval, then \( f(x_2) < f(x_1) \).

**Theorem 1**
(a) If \( f(x) \) is an increasing function that is differentiable, then \( f'(x) \geq 0 \).
(b) If \( f(x) \) is a decreasing function that is differentiable, then \( f'(x) \leq 0 \).

Conversely, we have the following result.

**Theorem 2**
(a) If \( f'(x) > 0 \) for all \( x \) in an interval, then \( f \) is an increasing function in that interval.
(b) If \( f'(x) < 0 \) for all \( x \) in an interval, then \( f \) is a decreasing function in that interval.

The proofs of these two theorems will not be given, but the results are intuitively obvious once one realizes that the derivative \( f'(x) \) is the slope of the tangent line of the curve \( y = f(x) \), and the function is increasing if the curve \( y = f(x) \) goes upward from left to right, while the function is decreasing if the curve goes downward from left to right.

**Example 1** Consider the function \( f(x) = x^2 \). When \( x_2 > x_1 \) we have \( f(x_2) - f(x_1) = x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1) \).
If \( x_1 > 0 \), then \( x_2 > 0 \) and \((x_2 + x_1)(x_2 - x_1)\) is the product of two positive numbers \( x_2 + x_1 \) and \( x_2 - x_1 \). Therefore,

\[
\frac{d}{dx}f(x) = (x^2)' = 2x
\]

Clearly, \( f'(x) = 2x \) is positive if \( x \) is positive and it is negative if \( x \) is negative. Thus by Theorem 2, we see that \( f(x) = x^2 \) is increasing for \( x \) in the interval \((0, \infty)\). □

This example shows clearly the advantage of using the derivative to analyse the function than using the definitions of increasing and decreasing functions.

**Example 2** Find the values of \( x \) for which the function

\[
f(x) = x^2 - 3x + 1
\]

is increasing or decreasing.

**Solution** We have

\[
\frac{df}{dx} = (x^2 - 3x + 1)' = 2x - 3
\]

\[
f'(x) = 2x - 3 > 0 \text{ when } 2x > 3, \text{ i.e., } x > \frac{3}{2}
\]

\[
f'(x) = 2x - 3 < 0 \text{ when } 2x < 3, \text{ i.e., } x < \frac{3}{2}
\]

Therefore the function is increasing when \( x > \frac{3}{2} \) and it is decreasing when \( x < \frac{3}{2} \). □
Note that \( y = x^2 - 3x + 1 \) represents a parabola, and it takes a minimum when 
\[
x = \frac{-b}{2a} = \frac{3}{2},
\]
where we view \( x^2 - 3x + 1 \) as a quadratic function \( ax^2 + bx + c \) with \( a = 1, b = -3 \) and \( c = 1 \).

**Example 3** For the cost function \( C(x) = 500 + 20x \) and the demand relation \( p = 100 - x \), find the ranges of \( x \) in which the cost function, revenue function, and profit function are increasing and decreasing functions.

**Solution**

\[
C'(x) = (500 + 20x)' = 20 > 0
\]
for all \( x \). Therefore the cost function is an increasing function for all values of \( x \).

The revenue function is

\[
R(x) = xp = x(100 - x) = 100x - x^2
\]
\[
R'(x) = (100x - x^2)' = 100 - 2x
\]

Hence

\[
R'(x) > 0 \text{ if } 100 - 2x > 0, \ i.e., \ 100 > 2x, 50 > x
\]
\[
R'(x) < 0 \text{ if } 100 - 2x < 0, \ i.e., 50 < x
\]

Therefore, the revenue is an increasing function of \( x \) if \( x < 50 \), but it is a decreasing function of \( x \) when \( x > 50 \).

The profit function is given by

\[
P(x) = R(x) - C(x) = 100x - x^2 - (500 + 20x)
\]
\[
= -x^2 + 80x - 500
\]

\[
P'(x) = (-x^2 + 80x - 500)' = -2x + 80
\]

\[
P'(x) > 0 \text{ if } -2x + 80 > 0, \ i.e., 80 > 2x, 40 > x
\]
\[
P'(x) < 0 \text{ if } -2x + 80 < 0, \ i.e., 40 < x
\]

Therefore the profit is an increasing function of \( x \) if \( x < 40 \), and it is a decreasing function of \( x \) if \( x > 40 \). \( \square \)
Many of the important applications of derivatives involve finding the maximum or minimum values of a particular function. For example, the profit a manufacturer makes depends on the price charged for the product, and the manufacturer is interested in knowing the price which makes his profit maximum. The optimum price (or best price) is obtained by a process called maximization or optimization of the profit function. In a similar way, a real estate company may be interested in knowing the rent to charge for the offices or apartments it controls to generate the maximum rental income.

Before we look at applications such as these, however, we shall discuss the theory of maxima and minima.

Definitions

(a) A function \( f(x) \) is said to have a local maximum at \( x = c \) if \( f(c) > f(x) \) for all \( x \) sufficiently near \( c \).

(b) A function \( f(x) \) is said to have a local minimum at \( x = c \) if \( f(c) < f(x) \) for all \( x \) sufficiently near \( c \).

(c) The term extremum is used to denote either a local maximum or a local minimum. Extrema is the plural of extremum.

A function may have more than one local maximum and more than one local minimum, as is shown in the diagram above, where the points \( A, C \) and \( E \) on the graph correspond to points where the function has local maxima, and the points \( B, D \) and \( F \) correspond to points where the function has local minima.

A local maximum or minimum value of a function is the \( y \)-coordinate at the point at which the graph has a local maximum or minimum. It is possible that a local minimum value is greater than a local maximum value of the same function.
**Definition** The value \( x = c \) is called a **critical point** for a continuous function \( f \) if \( f(c) \) is well-defined and if either \( f'(c) = 0 \) or \( f'(x) \) fails to exist at \( x = c \).

In the case when \( f'(c) = 0 \), the tangent to the graph of \( y = f(x) \) is horizontal at \( x = c \). When \( f'(c) \) fails to exist, the graph has a corner at \( x = c \) or the tangent to the graph becomes vertical at \( x = c \).

These possibilities are illustrated by the following diagrams.

It is clear from the above diagrams that the local extrema of a function \( f \) occur only at critical points. But not every critical point of a function corresponds to a local extremum. The following diagrams give three typical cases that critical points are not local extrema.

**Example 1** Determine the critical points of the function

\[
f(x) = \frac{(x - 1)^2}{x}
\]

**Solution**

\[
f'(x) = \left( \frac{(x - 1)^2}{x} \right)' = \frac{2(x - 1)x - (x - 1)^2}{x^2} = \frac{(x - 1)(2x - x + 1)}{x^2} = \frac{(x - 1)(x + 1)x^{-2}}
\]

\( f'(x) = 0 \) when \( x = 1 \) or \( x = -1 \), and \( f'(x) \) fails to exist when \( x = 0 \). \( \square \)
It is important to find a way to test when a critical point is a local maximum or local minimum. The simplest such test involves higher order derivatives.

Let \( y = f(x) \) be a given function of \( x \) with derivative \( \frac{dy}{dx} \). In full, we call this the **first derivative** of \( y \) with respect to \( x \). If \( f'(x) \) is a differentiable function of \( x \), its derivative is called the **second derivative** of \( y \) with respect to \( x \). If the second derivative is a differentiable function of \( x \), its derivative is called the **third derivative** of \( y \), and so on.

The first and all higher-order derivatives of \( y \) with respect to \( x \) are generally denoted by one of the following types of notation.

\[
\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^n y}{dx^n} \\
y', y'', y''' , \ldots, y^{(n)} \\
f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x)
\]

**Theorem** (Second Derivative Test). Let \( f(x) \) be twice differentiable at the critical point \( x = c \) (and hence \( f'(c) = 0 \)). Then

(a) \( x = c \) is a local maximum point of \( f \) when \( f''(c) < 0 \)

(b) \( x = c \) is a local minimum point of \( f \) when \( f''(c) > 0 \)

**Example 2** Find the local maximum and minimum values of

\[ f(x) = x^3 + 2x^2 - 4x - 8 \]

**Solution** We divide the solution into two steps. In step 1, we find all the critical points of \( f \). In step 2, we use the Second Derivative Test to check which critical points are local maximum or minimum point.

**Step 1**

\[
f'(x) = (x^3 + 2x^2 - 4x - 8)' \\
= 3x^2 + 4x - 4
\]

As the derivative exists for all values of \( x \), critical points are those values of \( x \) for which \( f'(x) = 0 \).
Letting $f'(x) = 0$, i.e., $3x^2 + 4x - 4 = 0$, we obtain

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(3)(-4)}}{2(3)} = \frac{-4 \pm \sqrt{16 + 48}}{6}$$

$$= \frac{-4 \pm \sqrt{64}}{6} = \frac{-4 \pm 8}{6}$$

$$= \frac{4}{6} \text{ or } \frac{-12}{6}$$

i.e. $x = \frac{2}{3}$ or $-2$

Hence there are two critical points, $x = \frac{2}{3}$ and $x = -2$.

**Step 2**

$$f''(x) = (3x^2 + 4x + 4)' = 6x + 4$$

When $x = \frac{2}{3}$,

$$f''(x) = f''\left(\frac{2}{3}\right) = 6\left(\frac{2}{3}\right) + 4 = 4 + 4 = 8 > 0$$

Therefore $x = \frac{2}{3}$ is a local minimum point, and the local minimum value is

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - f\left(\frac{2}{3}\right) - 8 = -\frac{256}{27}$$

When $x = -2$,

$$f''(x) = f''(-2) = 6(-2) + 4 = -12 + 4 = -8 < 0$$

Hence $x = -2$ is a local maximum point, and the local maximum value is

$$f(-2) = (-2)^3 + 2(-2)^2 - 4(-2) - 8 = 0$$

Thus, the only local maximum value of $f(x)$ is 0, and it occurs when $x = -2$; the only local minimum value is $-\frac{256}{27}$, and it occurs when $x = \frac{2}{3}$. □
31 Applications of Maxima and Minima

Many situations arise in practice when we want to maximize or minimize a certain quantity. The general rule is that we express this quantity as a function of a certain other quantity which is regarded as a variable, for example, the cost function is a function with variable \( x \), which stands for the number of items to be produced. The extremum of the function in general occurs at a critical point of the function, and we can use the second derivative test to confirm whether the critical point is a local minimum or local maximum point. We will explain the use of this general method through various examples.

**Example 1** (Maximizing Profits) A small manufacturing firm can sell all the items it can produce at a price of $6 each. The cost of producing \( x \) items per week (in dollars) is
\[
C(x) = 1000 + 6x - 0.003x^2 + 10^{-6}x^3.
\]
What value of \( x \) should be selected in order to maximize the profit?

**Solution** The revenue from selling \( x \) items at $6 each is \( R(x) = 6x \) dollars.

Therefore the profit per week is
\[
P(x) = R(x) - C(x)
\[
= 6x - (1000 + 6x - 0.003x^2 + 10^{-6}x^3)
\[
= -1000 + 0.003x^2 - 10^{-6}x^3
\]

To find \( x \) which maximizes \( P(x) \), we first find the critical points of \( P(x) \).
\[
P'(x) = (-1000 + 0.003x^2 - 10^{-6}x^3)'
\[
= 0.006x - 10^{-6}3x^2
\]

Letting \( P'(x) = 0 \) we obtain
\[
0.006x - 10^{-6} \cdot 3x^2 = 0
\]
\[
x(0.006 - 10^{-6} \cdot 3) = 0
\]

It gives \( x = 0 \) or \( x = \frac{0.006}{10^{-6} \cdot 3} = 2000 \).

Thus we have two critical points \( x = 0 \) and \( x = 2000 \). We now use the second derivative test to determine which critical point is a maximum point.
\[
P''(x) = (0.006x - 10^{-6} \cdot 3x^2)'
\]
Therefore $x = 2000$ maximizes $P(x)$, that is $x = 2000$ should be selected in order to maximize the profits.

**Example 2** (Price Decision) The cost of producing $x$ items per week is

$$C(x) = 1000 + 6x - 0.003x^2 + 10^{-6}x^3$$

For the particular item in question, the price at which $x$ items can be sold per week is given by the demand equation

$$p = 12 - 0.0015x$$

Determine the price and volume of sales at which the profit is maximum.

**Solution** The revenue per week is

$$R(x) = px = (12 - 0.0015x)x = 12x - 0.0015x^2$$

The profit is therefore given by

$$P(x) = R(x) - C(x)$$

$$= (12x - 0.0015x^2) - (1000 + 6x - 0.003x^2 + 10^{-6}x^3)$$

$$= -1000 + 6x + 0.0015x^2 - 10^{-6}x^3$$

To find when $P(x)$ has a maximum, we first set $P'(x) = 0$ to find the critical points.

$$P'(x) = 6 + 0.003x - (3 \times 10^{-6})x^2 = 0$$

To find $x$, we use the quadratic formula and obtain

$$x = \frac{-0.003 \pm \sqrt{0.003^2 - 4 \times 6 \times (-3 \times 10^{-6})}}{2(-3 \times 10^{-6})}$$

$$= \frac{-0.003 \pm \sqrt{0.00009 + 72 \times 10^{-6}}}{-6 \times 10^{-6}}$$

$$= \frac{-0.003 \pm \sqrt{0.00009 + 72 \times 10^{-6}}}{2000 \text{ or } -1000}$$

The negative solution has no practical significance, and when

$$x = 2000, \quad P''(x) = 0.003 - 6 \times 10^{-6}x$$

$$= 0.003 - 6 \times 10^{-6} \times 2000$$

$$= 0.003 - 0.012$$

$$= -0.009 < 0$$
Therefore $P(x)$ has a maximum when $x = 2000$. By the demand equation,

\[
p = 12 - 0.0015x
\]

\[
= 12 - 0.0015 \times 2000
\]

\[
= 9
\]

Thus the sales volume of 2000 items per week gives the maximum profit, and the price per item corresponding to this value of $x$ is 9 (dollars).

**Example 3** (Minimizing Cost) A tank is to be constructed with a horizontal, square base and vertical, rectangular sides. There is no top. The tank must hold 4 cubic meters of water. The material of which the tank is to be constructed costs $10 per square meter. What dimensions for the tank minimizes the cost of material?

**Solution** Let $x$ be the length of the sides and $y$ the height of the tank. Then the square base has area $x^2$, and the four rectangular sides each has area $xy$.

The total area of the material used is

\[
A = x^2 + 4xy
\]

The volume of the tank is $V = x^2y$ but by requirement, $V = 4$. Thus $x^2y = 4$ which gives $y = \frac{4}{x^2}$.

Substituting this into the expression of $A$ we obtain

\[
A = x^2 + 4xy = x^2 + 4x \left( \frac{4}{x^2} \right) = x^2 + \frac{16}{x}
\]

Now we can find for what value of $x$ the area $A$ has a minimum, and therefore the cost is minimized.

\[
\frac{dA}{dx} = \left( x^2 + \frac{16}{x} \right)' = 2x - \frac{16}{x^2}
\]
To find critical points of $A$ we let $\frac{dA}{dx} = 0$, i.e.

$$2x - \frac{16}{x^2} = 0, \quad 2x = \frac{16}{x^2}, \quad 2x^3 = 16, \quad x^3 = 8, \quad x = 2$$

There is one critical point $x = 2$. To use the second derivative test, we calculate

$$\frac{d^2 A}{dx^2} = \left(2x - \frac{16}{x^2}\right)'' = 2 + \frac{48}{x^3}$$

and at $x = 2$,

$$\frac{d^2 A}{dx^2} = 2 + \frac{48}{(2)^3} = 2 + \frac{48}{8} > 0$$

Therefore $A$ has a minimum when $x = 2$. From $y = \frac{4}{x^2}$ we deduce $y = \frac{4}{2^2} = 1$ when $x = 2$.

Thus the tank should have the sides each 2 meters in length and 1 meter in height to minimize the cost. \qed
32 Implicit Differentiation, Elasticity

A relationship between two variables is sometimes expressed through an implicit relation rather than by means of an explicit function. Thus instead of having $y$ given as a function $f(x)$ of the independent variable $x$, it is possible to have $x$ and $y$ related through an equation of the form $F(x, y) = 0$, in which both variables occur as arguments of some function $F$. For example, the equation

$$F(x, y) = x^3 + 2x^2y + y^3 - 1 = 0$$

expresses a certain relationship between $x$ and $y$, but $y$ is not given explicitly in terms of $x$.

The question we wish to consider in this lecture is how to calculate the derivative $\frac{dy}{dx}$ when $x$ and $y$ are related by an implicit equation. In certain cases, it is possible to solve the implicit equation $F(x, y) = 0$ and to obtain $y$ explicitly in terms of $x$. In such cases, the standard techniques of differentiation enable the derivative to be calculated in the usual way. However, in many cases it is not possible to solve for the explicit function, and to cope with such situations it is necessary that we use a new technique which is called **implicit differentiation**.

When using this technique, we differentiate each term in the given implicit relation with respect to the independent variable. This involves differentiating expressions involving $y$ with respect to $x$, and to do this, we make use of the chain rule. For example, suppose we wish to differentiate $y^3$ or $\ln y$ with respect to $x$. As $y$ is regarded as a function of $x$ (though not known explicitly), we have

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{d}{dx} (\ln y) = \frac{d}{dy} (\ln y)\frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}.$$

In general,

$$\frac{d}{dx} (f(y)) = f'(y)\frac{dy}{dx}.$$ 

**Example 1.** Find $\frac{dy}{dx}$ if $x^3 + y^3 = 1$.

**Solution** Differentiate both sides of the identity, we obtain

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(1).$$

The left hand side gives

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3x^2 + 3y^2\frac{dy}{dx}.$$
The right hand side equals 0. Hence
\[ 3x^2 + 3y^2 \frac{dy}{dx} = 0 \]
\[ 3y^2 \frac{dy}{dx} = -3x^2 \]
\[ \frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}. \]

In Example 1, we can also solve for \( y \) to obtain
\[ y^3 = 1 - x^3 \]
\[ y = (1 - x^3)^{\frac{1}{3}}. \]

Direct differentiation, using the chain rule, gives
\[ \frac{dy}{dx} = \left[ (1 - x^3)^{\frac{1}{3}} \right]' = \frac{1}{3}(1 - x^3)^{-\frac{2}{3}}(1 - x^3)' \]
\[ = \frac{1}{3}(1 - x^3)^{-\frac{2}{3}} \cdot (-3x^2) \]
\[ = -(1 - x^3)^{-\frac{2}{3}}x^2 \]
\[ = -\frac{x^2}{(1 - x^3)^{\frac{2}{3}}} \]

Recall \( y = (1 - x^3)^{\frac{1}{3}}. \) We find that
\[ -\frac{x^3}{(1 - x^3)^{\frac{2}{3}}} = -\frac{x^2}{y^2}. \]

Thus both methods yield the same result, though in different appearance.

Logarithmic Differentiation

With certain types of functions, a technique known as logarithmic differentiation can be used to ease the calculation of the derivative. We demonstrate the use of this method through some examples.

Example 2. Find \( \frac{dy}{dx} \) if \( y = \frac{(x^3 + 1)\sqrt{x^2 + 2}}{(x^2 + 1)^{\frac{3}{2}}} \).

Solution  We could use the quotient rule and product rule to start the differentiation. But let us start instead with taking the natural logarithm of both sides.
\[ \ln y = \ln \left[ \frac{(x^3 + 1)\sqrt{x^2 + 2}}{(x^2 + 1)^{\frac{3}{2}}} \right] \]
\[ = \ln(x^3 + 1) + \ln \sqrt{x^2 + 2} - \ln \left[ (x^2 + 1)^{\frac{3}{2}} \right] \]
\[ = \ln(x^3 + 1) + \frac{1}{2} \ln(x^2 + 2) - \frac{1}{3} \ln(x^2 + 1). \]
Now taking the derivative with respect to \( x \) to both sides of the identity
\[
\ln y = \ln(x^3 + 1) + \frac{1}{2} \ln(x^2 + 2) - \frac{1}{3} \ln(x^2 + 1),
\]
we obtain
\[
\frac{d}{dx}(\ln y) = \left[\ln(x^3 + 1)\right]' + \frac{1}{2} \left[\ln(x^2 + 2)\right]' - \frac{1}{3} \left[\ln(x^2 + 1)\right]'
\]
\[
\frac{1}{y} \frac{dy}{dx} = \frac{3x^2}{x^3 + 1} + \frac{1}{2} \frac{2x}{x^2 + 2} - \frac{1}{3} \frac{2x}{x^2 + 1}
\]
\[
\frac{dy}{dx} = y \left[ \frac{3x^2}{x^3 + 1} + \frac{x}{x^2 + 2} - \frac{2}{3} \frac{x}{x^2 + 1} \right]
\]
\[
= \frac{(x^3 + 1)\sqrt{x^2 + 2}}{(x^2 + 1)^{\frac{3}{2}}} \left( \frac{3x^2}{x^3 + 1} + \frac{x}{x^2 + 2} - \frac{2}{3} \frac{x}{x^2 + 1} \right).
\]

\[\Box\]

**Example 3** Find \( y' \) where \( y = x^x \).

**Solution** Taking logarithms of both sides of the equation
\[
y = x^x,
\]
we obtain
\[
\ln y = \ln(x^x) = x \ln x.
\]
Hence
\[
\frac{d}{dx}(\ln y) = (x \ln x)' = x(\ln x)' + (x)' \ln x
\]
\[
= \frac{1}{x} \ln x + 1 \ln x
\]
\[
= 1 + \ln x.
\]
But \( \frac{d}{dx}(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx} \).

Therefore
\[
\frac{1}{y} \frac{dy}{dx} = 1 + \ln x
\]
\[
\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x).
\]

\[\Box\]

**Elasticity**

A concept widely used in economics is that of elasticity. We shall introduce this idea via the so-called **elasticity of demand**.
Let the demand law be given by

\[ x = f(p) \]

where \( x \) is the number of units that can be sold at price \( p \) per unit. The elasticity of demand is usually represented by the Greek letter \( \eta \) (read eta) and is defined as follows:

\[
\eta = \frac{p}{x} \frac{dx}{dp} = \frac{p}{x} f'(p) = \frac{p f'(p)}{f(p)}
\]

Recall that \( \frac{dx}{dp} \) is approximately the average rate of change \( \frac{\Delta x}{\Delta p} \), when \( \Delta p \) is small where \( \Delta x = f(p + \Delta p) - f(p) \). Therefore,

\[
\eta \approx \frac{p}{x} \frac{\Delta x}{\Delta p} = \frac{\Delta x}{x} \frac{\Delta p}{\Delta p} = \frac{\Delta x}{x} \frac{\Delta p}{p}
\]

\( \frac{\Delta x}{x} \) represents the percentage change in demand, for example, \( \frac{\Delta x}{x} = 0.1 \) would mean the change in demand is 10%. Similarly, \( \frac{\Delta p}{p} \) represents the percentage change in price. Therefore,

Percentage Change in Demand \( \approx (\text{elasticity of demand}) \times (\text{percentage change in price}) \)

**Example 4.** Calculate the elasticity of demand if \( x = 500(10 - p) \) for \( p = 4 \).

**Solution**

\[
\frac{dx}{dp} = [500(10 - p)]' = 500(10 - p)' = 500(-1) = -500
\]

\[
\eta = \frac{p}{x} \frac{dx}{dp} = \frac{p}{500(10 - p)} (-500) = \frac{-500p}{500(10 - p)} = \frac{-p}{10 - p}.
\]

When \( p = 4 \).

\[
\eta = \frac{-4}{10 - 4} = \frac{-4}{6} = -\frac{2}{3}.
\]
The idea of elasticity can be used for any pair of variables that are related by a function. If \( y = f(x) \), then the \textbf{elasticity of} \( y \) \textbf{with respect to} \( x \) is defined as

\[
\eta = \frac{x \, dy}{y \, dx}.
\]

Note that

\[
\frac{d}{dx}(\ln y) = \frac{1 \, dy}{y \, dx}.
\]

Therefore, \( \eta = x \frac{d}{dx}(\ln y) \).
33 Functions of Several Variables

In most practical problems, the functions usually have more than one variable. Therefore, it is important to know how to differentiate functions of several variables.

Let \( z = f(x, y) \) be a function of two independent variables. If the variable \( y \) is held fixed at a value \( y = y_0 \), then the relation \( z = f(x, y_0) \) expresses \( z \) as a function of the one variable \( x \), and hence we can differentiate \( z \) with respect to \( x \) from the relation \( z = f(x, y_0) \). For example, if \( f(x, y) = x^2y + y^2 \) and \( y_0 = 1 \), then \( f(x, y_0) = x^2(1) + (1)^2 = x^2 + 1 \), and the derivative with respect to \( x \) is

\[
(x^2 + 1)' = 2x.
\]

Recall that by definition,

\[
\frac{d}{dx} f(x, y_0) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y_0) - f(x, y_0)}{\Delta x}
\]

Regarding \( f \) as a two variable function, this derivative is called the partial derivative of \( z \) with respect to \( x \), and is usually denoted by \( \frac{\partial z}{\partial x} \). (Note that we use \( \partial \) not \( d \) in this situation. The letter \( d \) is reserved for the derivative of a function of a single variable.)

A formal definition of partial derivatives is given below.

**Definition** Let \( z = f(x, y) \) be a function of \( x \) and \( y \). Then the **partial derivative of \( z \) with respect to \( x \)** is defined to be

\[
\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.
\]

In writing this definition we have dropped the subscript from \( y_0 \), but we must remember that when calculating \( \frac{\partial z}{\partial x} \), the variable \( y \) is held constant.

Correspondingly, the **partial derivative of \( z \) with respect to \( y \)** is defined to be

\[
\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.
\]

In calculating \( \frac{\partial z}{\partial y} \), the variable \( x \) is held constant and the differentiation is carried out with respect to \( y \).

Partial derivatives can be evaluated using essentially the same techniques as those used for evaluating ordinary derivatives. We must simply remember to treat any variable except the one with which we are differentiating as if it were a constant.
Apart from this, the familiar power formula, product and quotient rules, and the chain rule can all be used in the usual way.

**Example 1.** Calculate \( \frac{\partial z}{\partial y} \) when \( z = x^3 + 2x^2y + y^4 + 6 \).

**Solution** Treating \( x \) as a constant and differentiating with respect to \( y \), we obtain

\[
\frac{\partial z}{\partial y} = 0 + 2x^2(1) + 4y^3 + 0 = 2x^2 + 4y^3.
\]

\( \square \)

**Example 2.** Find \( \frac{\partial z}{\partial x} \) for \( z = \sqrt{x^2 + y^2} \).

**Solution** With \( y \) held constant, we use the chain rule.

\[
\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2)^{\frac{1}{2}} = \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \frac{\partial}{\partial x} (x^2 + y^2)
\]

\[
= \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} 2x = \frac{x}{\sqrt{x^2 + y^2}}.
\]

\( \square \)

**Example 3.** Calculate \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) for \( z = \frac{x^3 + y^3}{\ln y} \).

**Solution.** When we calculate \( \frac{\partial z}{\partial x} \), \( y \) is held constant. Therefore, \( \ln y \) is treated as a constant. We differentiate with respect to \( x \), and obtain

\[
\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left( \frac{x^3 + y^3}{\ln y} \right) = \frac{\partial}{\partial x} \left[ \left( \frac{1}{\ln y} \right) \cdot (x^3 + y^3) \right]
\]

\[
= \frac{3x^2}{\ln y} \cdot \frac{\partial}{\partial x} (x^3 + y^3) = \frac{1}{\ln y} (3x^2 + 0) = \frac{3x^2}{\ln y}.
\]

To calculate \( \frac{\partial z}{\partial y} \), then \( x \) is held constant and we must use the quotient rule.

\[
\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left( \frac{x^3 + y^3}{\ln y} \right) = \frac{\frac{\partial}{\partial y} (x^3 + y^3) \cdot (\ln y) - (x^3 + y^3) \frac{\partial}{\partial y} (\ln y)}{(\ln y)^2}
\]

\[
= \frac{(3y^2)(\ln y) - (x^3 + y^3) \cdot \left( \frac{1}{y} \right)}{(\ln y)^2} = \frac{3y^3 \ln y - (x^3 + y^3)}{y(\ln y)^2}
\]

\( \square \)

It can be seen from these examples that the calculation of partial derivatives of functions of two variables is essentially not different from differentiating a function
of one variable. We simply treat one variable as a constant and differentiate in the familiar way.

The partial derivatives \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) themselves are functions of \( x \) and \( y \), and therefore we can construct their partial derivatives with respect to \( x \) and \( y \). These are called **second-order partial derivatives** of \( z \), and the following notations are used:

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right), \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right),
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right), \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right).
\]

The two derivative \( \frac{\partial^2 z}{\partial y \partial x} \) and \( \frac{\partial^2 z}{\partial x \partial y} \) are often called **mixed partial derivatives** of second order. It can be proved that these mixed derivatives are equal provided that they are continuous functions of \( x \) and \( y \).

**Example 4.** Calculate all the second-order partial derivatives of the function

\[ z = x^3 y^2 + e^y. \]

**Solution** We need to start with the first-order partial derivatives

\[
\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left( x^3 y^2 \right) + \frac{\partial}{\partial x} (e^y) = 3x^2 y^2 + 0 = 3x^2 y^2
\]

\[
\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left( x^3 y^2 \right) + \frac{\partial}{\partial y} (e^y) = x^3(2y) + e^y = 2x^3 y + e^y.
\]

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (3x^2 y^2) = 6xy^2
\]

\[
\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (3x^2 y^2) = 3x^2(2y) = 6x^2 y
\]

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left( 2x^3 y + e^y \right) = 2x^3 + e^y
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( 2x^3 y + e^y \right) = 6x^2 y
\]

Note that in example 4, \( \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 6x^2 y. \)

If we take partial derivatives of second-order partial derivatives, we obtain third-order partial derivatives, and so on.

\[
\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x} \left( \frac{\partial^2 z}{\partial x^2} \right), \quad \frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial^2 z}{\partial x \partial y} \right),
\]

\[
\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial^2 z}{\partial y^2} \right).
\]
 Provided that all the partial derivatives of the given order are continuous, the order in which the $x$ and $y$ differentiations are carried out does not affect the final result. Thus
\[
\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial x \partial x \partial y}
\]

**Example 5** Calculate $\frac{\partial^3 z}{\partial x^2 \partial y}$ for $z = x^5 y^6 + x \ln y$.

**Solution**
\[
\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^5 y^6 + x \ln y)
= x^5(6y^5) + x \left( \frac{1}{y} \right)
= 6x^5 y^5 + \frac{x}{y}
\]
\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)
= \frac{\partial}{\partial x} \left( 6x^5 y^5 + \frac{x}{y} \right)
= 6(5x^4) y^5 + \frac{1}{y}
= 30x^4 y^5 + \frac{1}{y}
\]
\[
\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial^2 z}{\partial x \partial y} \right)
= \frac{\partial}{\partial x} \left( 30x^4 y^5 + \frac{1}{y} \right)
= 30 \left( 4x^3 \right) y^5 + 0
= 120x^3 y^5.
\]
34 Applications of Partial Derivatives

The ordinary derivative \( \frac{dy}{dx} \) can be regarded as the rate of change of \( y \) with respect to \( x \). This interpretation is often useful. For example, the marginal revenue \( R'(x) \) gives the rate of change of revenue with respect to the volume of sales, or, approximately, the change in revenue per additional unit sold. Similar interpretations can be made in the case of partial derivatives. For example, if \( z = f(x, y) \), then \( \frac{\partial z}{\partial x} \) gives the rate of change of \( z \) with respect to \( x \) when \( y \) is fixed.

Example 1 A new product is launched onto the market. The volume of sales \( x \) increases as a function of time \( t \) and also depends on the amount \( A \) spent on the advertising campaign. If, with \( t \) measured in months and \( A \) in dollars,

\[
x = 200(5 - e^{-0.002A})(1 - e^{-t})
\]

calculate \( \frac{\partial x}{\partial t} \) and \( \frac{\partial x}{\partial A} \). Evaluate these partial derivatives when \( t = 1 \) and \( A = 400 \) and interpret them.

Solution We have

\[
\frac{\partial x}{\partial t} = 200(5 - e^{-0.002A})e^{-t}
\]
\[
\frac{\partial x}{\partial A} = 200(0.002e^{-0.002A})(1 - e^{-t})
\]
\[
= 0.4e^{-0.002A}(1 - e^{-t})
\]

Setting \( t = 1 \) and \( A = 400 \), we obtain

\[
\frac{\partial x}{\partial t} = 200(5 - e^{-0.002\times400})e^{-1} \approx 335
\]
\[
\frac{\partial x}{\partial A} = 0.4e^{-0.002\times400}(1 - e^{-1}) \approx 0.11
\]

The partial derivative \( \frac{\partial x}{\partial t} \) gives the rate of increase in the sales volume with respect to time when the advertising expenditure is maintained fixed. When this expenditure is fixed at $400, the volume of sales after one month \( (t = 1) \) is growing at the instantaneous rate of 335 per month.

Similarly, \( \frac{\partial x}{\partial A} \) gives the rate of change in the sales volume at a fixed time with respect to the expenditure on advertising. At the time \( t = 1 \), when $400 is already spent on advertising, an additional dollar so spent will increase the sales volume by 0.11 unit.

Example 2 The production function of a certain firm is given by

\[
P = 5L + 2L^2 + 3LK + 8K + 3K^2
\]
where \( L \) is the labor input measured in thousands of work-hours per week, \( K \) is the cost of capital investment measured in thousands of dollars per week, and \( P \) is the weekly production in hundreds of items. \( \frac{\partial P}{\partial L} \) is called the marginal productivity of labor and \( \frac{\partial P}{\partial K} \) is called the marginal productivity of capital. Determine the marginal productivities when \( L = 5 \) and \( K = 12 \) and interpret the result.

**Solution** The marginal productivity of labor is given by

\[
\frac{\partial P}{\partial L} = \frac{\partial}{\partial L}(5L + 2L^2 + 3LK + 8K + 3K^2) = 5 + 4L + 3K
\]

The marginal productivity of capital equals

\[
\frac{\partial P}{\partial K} = \frac{\partial}{\partial K}(5L + 2L^2 + 3LK + 8K + 3K^2) = 3L + 8 + 6K
\]

When \( L = 5 \) and \( K = 12 \),

\[
\frac{\partial P}{\partial L} = 5 + 4(5) + 3(12) = 61 \\
\frac{\partial P}{\partial K} = 3(5) + 8 + 6(12) = 95
\]

This means that when \( L = 5 \) and \( K = 12 \) (that is, 5000 work-hours per week are used and the cost of capital investment is $12,000 per week), then the production \( P \) increases by 61 (hundred items) for each unit (thousand work-hours per week) increases in \( L \), and \( P \) increases by 95 for each unit increase in \( K \). □

**Optimization**

A very important application of partial derivatives is to find the maximum or minimum of a function of several variables.

**Definition** The function \( f(x, y) \) has a **local maximum** at the point \((x_0, y_0)\) if \( f(x, y) < f(x_0, y_0) \) for all points \((x, y)\) close to \((x_0, y_0)\), except for \((x_0, y_0)\) itself.

The function \( f(x, y) \) has a **local minimum** at the point \((x_0, y_0)\) if \( f(x, y) > f(x_0, y_0) \) for all points \((x, y)\) close to \((x_0, y_0)\) except for \((x_0, y_0)\) itself.

Clearly, if \( f(x, y) \) has a local maximum at \((x_0, y_0)\), then the one variable function \( g(x) = f(x, y_0) \) has a local maximum at \( x = x_0 \).

It follows from an earlier theorem that \( g'(x_0) = 0 \) when this derivative exists. But \( g'(x_0) = f_x(x_0, y_0) \), where we use \( f_x(x_0, y_0) \) to denote the partial derivative
\( \frac{\partial z}{\partial x} \) evaluated at \((x_0, y_0)\) and \(z = f(x, y)\). Hence \(f_x(x_0, y_0) = 0\) when this partial derivative exists. Similarly, the one variable function \(h(y) = f(x_0, y)\) has a local maximum at \(y = y_0\), and this implies \(h'(y_0) = 0\) when this derivative exists. But \(h'(y_0) = f_y(x_0, y_0)\). Thus \(f_y(x_0, y_0) = 0\) when the partial derivative exists.

If \(f(x, y)\) has a local minimum at \((x_0, y_0)\), a similar consideration shows \(f_x(x_0, y_0) = f_y(x_0, y_0) = 0\) provided that both partial derivatives exist.

Thus we are led to the following theorem.

**Theorem** If \(f(x, y)\) has a local maximum or a local minimum at the point \((x_0, y_0)\), then

\[
f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0
\]

provided that both partial derivatives exist.

**Definition** A **critical point** of a smooth function \(f(x, y)\) is a point \((x_0, y_0)\) at which \(f_x(x_0, y_0) = f_y(x_0, y_0) = 0\).

Thus the above theorem says that local maximum or minimum of \(f(x, y)\) occurs at a critical point of the function. However, there might be critical points which are not local maximum nor local minimum point.

**Example 3** Find the critical points of the function

\[
f(x, y) = x^3 + x^2 y + x - y
\]

**Solution** We must set the two partial derivatives \(f_x\) and \(f_y\) equal to zero:

\[
f_x(x, y) = 3x^2 + 2xy + 1 = 0
\]
\[
f_y(x, y) = x^2 - 1 = 0
\]

Critical points are solutions of this equation system. From the second equation we obtain

\[x^2 = 1, \quad x = \pm 1\]

Substituting \(x = 1\) into the first equation, we obtain

\[3(1)^2 + 2(1)y + 1 = 0, \quad 4 + 2y = 0, \quad 2y = -4, \quad y = -2\]

Hence \((x, y) = (1, -2)\) is a critical point.
Substituting $x = -1$ into the first equation, we obtain

$$3(-1)^2 + 2(-1)y + 1 = 0, \quad 4 - 2y = 0, \quad 2y = 4, \quad y = 2$$

Therefore $(x, y) = (-1, 2)$ is another critical point. As these are the only two pairs of solutions for the system of equations obtained from setting $f_x = 0$ and $f_y = 0$, there is no other critical point. $\Box$
35 Optimization for Two Variable Functions

We learned in the last lecture that if $f(x, y)$ has a local maximum or local minimum at $(x_0, y_0)$, then $(x_0, y_0)$ is a critical point of $f$ provided that both partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

To see that a critical point need not be a local maximum or local minimum point, let us look at the function $f(x, y) = x^2 - y^2$. We have

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2 - y^2) = 2x,$$

$$f_y(x, y) = \frac{\partial}{\partial y}(x^2 - y^2) = -2y.$$

Therefore $f_x(0, 0) = 0$, $f_y(0, 0) = 0$, that is, $(0, 0)$ is a critical point. For all the points of the form $(x, 0)$ close to $(0, 0)$,

$$f(x, 0) = x^2 > f(0, 0) = 0$$

but for all the points of the form $(0, y)$ close to $(0, 0)$,

$$f(0, y) = -y^2 < f(0, 0) = 0.$$

Therefore, $f(x, y)$ has neither a local maximum nor a local minimum at $(0, 0)$. This example shows that a critical point may not be a local extremum point.

To determine when a critical point is a local maximum or local minimum point, we have the following second derivative test for two variable functions.

Theorem. Let $(x_0, y_0)$ be a critical point of $f(x, y)$, i.e., $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Let

$$\Delta(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2.$$

(a) If $f_{xx}(x_0, y_0) < 0$, $f_{yy}(x_0, y_0) < 0$ and $\Delta(x_0, y_0) > 0$, then $f(x, y)$ has a local maximum at $(x_0, y_0)$.

(b) If $f_{xx}(x_0, y_0) > 0$, $f_{yy}(x_0, y_0) > 0$ and $\Delta(x_0, y_0) > 0$, then $f(x, y)$ has a local minimum at $(x_0, y_0)$.

(c) If $\Delta(x_0, y_0) < 0$, then $(x_0, y_0)$ is not a local extremum point of $f(x, y)$. 
The proof of this theorem is beyond the scope of this unit, but we should know how to use the theorem.

In the example we looked above, we checked directly that the critical point \((0, 0)\) of \(f(x, y) = x^2 - y^2\) is not a local extremum point. Let us now use the second derivative test to recover this. We have

\[
\begin{align*}
  f_{xx}(x, y) &= \frac{\partial}{\partial x}(f_x(x, y)) = \frac{\partial}{\partial x}(2x) = 2 \\
  f_{yy}(x, y) &= \frac{\partial}{\partial y}(f_y(x, y)) = \frac{\partial}{\partial y}(-2y) = -2 \\
  f_{xy}(x, y) &= \frac{\partial}{\partial y}(f_x(x, y)) = \frac{\partial}{\partial y}(2x) = 0.
\end{align*}
\]

Therefore,

\[
\triangle(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(-2) - 0^2 = -4 < 0.
\]

By the second derivative test, \((0, 0)\) is not a local extremum point.

**Example 1.** Find the local extrema of the function

\[f(x, y) = x^3 + x^2y + x - 4y\]

**Solution.** We first find all the critical points and then use the second derivative test to determine which are local extrema.

To find critical points, we set \(f_x\) and \(f_y\) equal to zero:

\[
\begin{align*}
  f_x(x, y) &= \frac{\partial}{\partial x}(x^3 + x^2y + x - 4y) = 3x^2 + 2xy + 1 = 0 \\
  f_y(x, y) &= \frac{\partial}{\partial y}(x^3 + x^2y + x - 4y) = x^2 - 4 = 0.
\end{align*}
\]

From the second equation we obtain

\[x^2 = 4, \quad x = \pm 2.\]

Substitute \(x = 2\) into the first equation. We obtain

\[
\begin{align*}
  3(2)^2 + 2(2)y + 1 &= 0 \\
  13 + 4y &= 0 \\
  y &= -\frac{13}{4}.
\end{align*}
\]

Hence \((x, y) = (2, -\frac{13}{4})\) is a critical point.

Substituting \(x = -2\) into the first equation we obtain
\[ 3(-2)^2 + 2(-2)y + 1 = 0 \]
\[ 13 - 4y = 0 \]
\[ y = \frac{13}{4} \]

Thus \((x, y) = (-2, \frac{13}{4})\) is another critical point.

Summarizing, we have two critical points, \(\left(2, -\frac{13}{4}\right)\) and \(\left(-2, \frac{13}{4}\right)\).

Now we use the second derivative test.

\[
\begin{align*}
  f_{xx}(x, y) &= \frac{\partial}{\partial x}(f_x(x, y)) = \frac{\partial}{\partial x}(3x^2 + 2xy + 1) = 6x + 2y \\
  f_{yy}(x, y) &= \frac{\partial}{\partial y}(f_y(x, y)) = \frac{\partial}{\partial y}(x^2 - 4) = 0 \\
  f_{xy}(x, y) &= \frac{\partial}{\partial y}(f_x(x, y)) = \frac{\partial}{\partial y}(3x^2 + 2xy + 1) = 2x
\end{align*}
\]

Hence

\[
\begin{align*}
  \triangle(x, y) &= f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\
  &= (6x + 2y)(0) - (2x)^2 \\
  &= -4x^2.
\end{align*}
\]

When \((x, y) = \left(2, -\frac{13}{4}\right)\), we have

\[
\triangle \left(2, -\frac{13}{4}\right) = -4(-2)^2 = -16 < 0
\]

When \((x, y) = \left(-2, \frac{13}{4}\right)\), we obtain

\[
\triangle \left(-2, \frac{13}{4}\right) = -4(-2)^2 = -16 < 0.
\]

Therefore no critical point is a local extremum. We conclude that the function has no local extremum. \(\square\).

**Example 2.** A company produces toothpaste in two sizes, 100 milliliter and 150 milliliter. The costs of producing for each size tube are 60 cents and 90 cents, respectively. The weekly demands \(x_1\) and \(x_2\) (in thousands) for the two sizes are

\[
x_1 = 3(p_2 - p_1), \quad x_2 = 320 + 3p_1 - 5p_2
\]

where \(p_1\) and \(p_2\) are the prices in cents per tube. Determine the prices \(p_1\) and \(p_2\) that will maximize the company’s profits.
Solution. The profit obtained from each 100-milliliter tube of toothpaste is \((p_1 - 60)\) cents, and the profit from each 150-milliliter tube is \((p_2 - 90)\) cents. Therefore the profit \(P\) (in thousands of cents, because the demands are in thousands) obtained by selling \(x_1\) tubes of 100-milliliter size and \(x_2\) 150-milliliter size is given by

\[
P = (p_1 - 60)x_1 + (p_2 - 90)x_2.
\]

We substitute the relations

\[
x_1 = 3(p_2 - p_1), \ x_2 = 320 + 3p_1 - 5p_2
\]

into the expression of \(P\) and obtain \(P\) as a two variable function with variables \(p_1\) and \(p_2\):

\[
P &= 3(p_1 - 60)(p_2 - p_1) + (p_2 - 90)(320 + 3p_1 - 5p_2) \\
&= -3p_1^2 - 5p_2^2 + 6p_1p_2 - 90p_1 + 590p_2 - 28800
\]

To find when \(P\) has a local maximum, we follow the steps in the solution of example 1, namely, first find the critical points and then use the second derivative test.

\[
\frac{\partial P}{\partial p_1} = \frac{\partial}{\partial p_1} (-3p_1^2 - 5p_2^2 + 6p_1p_2 - 90p_1 + 590p_2 - 28800)
\]

\[
= -6p_1 + 6p_2 - 90
\]

\[
\frac{\partial P}{\partial p_2} = \frac{\partial}{\partial p_2} (-3p_1^2 - 5p_2^2 + 6p_1p_2 - 90p_1 + 590p_2 - 28800)
\]

\[
= -10p_2 + 6p_1 + 590.
\]

To find the critical points, we let \(\frac{\partial P}{\partial p_1} = \frac{\partial P}{\partial p_2} = 0\) to obtain

\[
-6p_1 + 6p_2 - 90 = 0 \\
-10p_2 + 6p_1 + 590 = 0
\]

Adding the two equations we obtain

\[
-4p_2 + 500 = 0, \quad p_2 = \frac{500}{4} = 125.
\]

Substitute \(p_2 = 125\) into the first equation we have

\[
-6p_1 + 6(125) - 90 = 0 \\
-6p_1 + 750 - 90 = 0 \\
p_1 = \frac{660}{6} = 110.
\]
Thus there is one critical point, namely \((p_1, p_2) = (110, 125)\).

To use the second derivative test, we calculate

\[
\frac{\partial^2 P}{\partial p_1^2} = \frac{\partial}{\partial p_1} \left( \frac{\partial P}{\partial p_1} \right) = \frac{\partial}{\partial p_1} (-6p_1 + 6p_2 - 90) = -6
\]

\[
\frac{\partial^2 P}{\partial p_2^2} = \frac{\partial}{\partial p_2} \left( \frac{\partial P}{\partial p_2} \right) = \frac{\partial}{\partial p_2} (-10p_2 + 6p_1 + 590) = -10
\]

\[
\frac{\partial^2 P}{\partial p_1 \partial p_2} = \frac{\partial}{\partial p_1} \left( \frac{\partial P}{\partial p_2} \right) = \frac{\partial}{\partial p_1} (-10p_2 + 6p_1 + 590) = 6
\]

Therefore

\[
\Delta = \frac{\partial^2 P}{\partial p_1^2} \cdot \frac{\partial^2 P}{\partial p_2^2} - \left( \frac{\partial^2 P}{\partial p_1 \partial p_2} \right)^2
\]

\[
= (-6)(-10) - 6^2 = 60 - 36 = 24 > 0.
\]

As \( \frac{\partial^2 P}{\partial p_1^2} = -60 < 0, \frac{\partial^2 P}{\partial p_2^2} = -10 < 0, \) we conclude that \( P \) has a local maximum at \((p_1, p_2) = (110, 125)\). Thus the prices \( p_1 = 110 \) cents and \( p_2 = 125 \) cents yield a maximum profit for the company. \(\square\).
36 Antiderivatives

So far in our study of calculus, we have been concerned with the process of differentiation - that is, the calculation and use of the derivatives of functions. This part of the subject is called differential calculus. We shall now turn briefly to the second area of study within the general area of calculus, called integral calculus, in which we are concerned with the process opposite to differentiation.

We have seen before that, if \( s(t) \) is the distance traveled in time \( t \) by a moving object, then the instantaneous velocity is \( v(t) = s'(t) \), the derivative of \( s(t) \). In order to calculate \( v \), we simply differentiate \( s(t) \). However, it may happen that we already know the velocity function \( v(t) \) and wish to calculate the distance \( s \) traveled. In such a situation, we know the derivative \( s'(t) \) and require the function \( s(t) \), a step opposite to that of differentiation.

The process of finding the function when its derivative is given is called integration, and the function to be found is called the antiderivative or the integral of the given function.

To evaluate the antiderivative of some function \( f(x) \), we must find a function \( F(x) \) whose derivative is \( f(x) \). For example, if \( f(x) = 4x^3 \), then since we know \( (x^4)' = 4x^3 \), we can conclude that \( F(x) = x^4 \) is an antiderivative of \( f(x) = 4x^3 \).

However, this answer is not unique. If \( C \) is any constant, then the function \( x^4 + C \) is also an antiderivative of \( 4x^3 \). Such a constant \( C \), which can have any arbitrary value, is called the constant of integration. It can be easily proved that if \( F(x) \) is any antiderivative of a given function \( f(x) \), then any other antiderivative of \( f(x) \) differs form \( F(x) \) only by a constant. Therefore, if \( F(x) \) is a particular antiderivative of \( f(x) \), then the general antiderivative of \( f(x) \) is given by \( F(x) + C \), where \( C \) is an arbitrary constant. If follows that the general antiderivative of \( 4x^3 \) is \( x^4 + C \).

Since the constant of integration is arbitrary, the integral (or antiderivative) is given the more complete name indefinite integral. The expression

\[
\int f(x) \, dx
\]

is used to denote an arbitrary member of the set of antiderivatives of \( f \). It is read as the integral of \( f(x), \, dx \). In such an expression, the function \( f(x) \) is called the integrand and the symbol \( \int \) is the integral sign. The integral sign and \( dx \) go together; \( dx \) specifies that the variable of integration is \( x \).

If \( F(x) \) is one particular antiderivative of \( f(x) \), then

\[
\int f(x) \, dx = F(x) + C
\]
where $C$ is an arbitrary constant. For example,

$$\int 4x^3 \, dx = x^4 + C.$$ 

From the definition of integral, it is clear that

$$\frac{d}{dx} \left( \int f(x) \, dx \right) = f(x)$$

From the formulas

$$(x^{n+1})' = (n+1)x^n,$$

$$(e^x)' = e^x,$$

$$(\ln x)' = \frac{1}{x},$$

we immediately obtain

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int x^{-1} \, dx = \ln x + C$$

$$\int e^x \, dx = e^x + C$$

When $x < 0$, we know $\ln(x)$ is defined and

$$\frac{d}{dx} \ln(-x) = \frac{1}{(-x)}(-x)' = \frac{1}{-x}(-1) = \frac{1}{x}$$

Therefore, we have

$$\frac{d}{dx} \ln |x| = \frac{1}{x} \quad (x \neq 0).$$

It follows that

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

The following properties of integrals follow directly from the corresponding properties of derivatives.
1. \[ \int cf(x) \, dx = c \int f(x) \, dx, \] \( c \) is a constant.

2. \[ \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \]

3. \[ \int [f(x) - g(x)] \, dx = \int f(x) \, dx - \int g(x) \, dx \]

**Example 1** Find \( \int (6x^3 + 7x + 1) \, dx \)

**Solution**
\[
\int (6x^3 + 7x + 1) \, dx = 6 \int x^3 \, dx + 7 \int x \, dx + \int 1 \, dx \\
= 6 \cdot \frac{x^4}{4} + 7 \cdot \frac{x^2}{2} + \frac{x^{1+0}}{1+0} + C \\
= \frac{3}{2} x^4 + \frac{7}{2} x^2 + x + C.
\]

**Example 2.** Find \( \int \left( t + \frac{2}{t} \right)^2 \, dt \).

**Solution**
\[
\left( t + \frac{2}{t} \right)^2 = t^2 + 2 \cdot t \cdot \frac{2}{t} + \left( \frac{2}{t} \right)^2 \\
= t^2 + 4 + 4t^{-2}
\]

Hence
\[
\int \left( t + \frac{2}{t} \right)^2 \, dt = \int \left( t^2 + 4 + 4t^{-2} \right) \, dt \\
= \int t^2 \, dt + 4 \int 1 \, dt + 4 \int t^{-2} \, dt \\
= \frac{t^3}{3} + 4t + 4t^{-1} + C = \frac{1}{3} t^3 + 4t - 4t^{-1} + C.
\]

**Example 3.** The marginal revenue of a firm is given by
\[
R'(x) = 15 - 0.01x.
\]

(a) Find the revenue function.

(b) Find the demand relation for the firm’s product.
Solution

(a) The revenue function $R(x)$ is the integral of the marginal revenue function. Thus

$$R(x) = \int R'(x) \, dx = \int (15 - 0.01x) \, dx$$

$$= 15x - 0.01 \frac{x^2}{2} + C$$

$$= 15x - 0.005x^2 + C,$$

where $C$ is the constant of integration. When $x = 0$, $R(0)$ should be 0. But clearly $R(0) = C$. Hence $C$ should be 0. Thus

$$R(x) = 15x - 0.005x^2.$$

(b) If $p$ stands for the price per item, then

$$R(x) = px.$$

Hence

$$px = 15x - 0.005x^2.$$

It follows that

$$p = 15 - 0.005x,$$

which is the required demand relation. $\Box$
37 Definite Integrals

Definition Let \( f(x) \) be a function with an antiderivative that we denote by \( F(x) \). Let \( a \) and \( b \) be any two real numbers such that \( f(x) \) and \( F(x) \) exist for all values of \( x \) in the closed interval \([a, b]\). The definite integral of \( f(x) \) from \( x = a \) to \( x = b \) is denoted by \( \int_{a}^{b} f(x) \, dx \) and is defined by

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

The numbers \( a \) and \( b \) are called the limits of integration, \( a \) the lower limit and \( b \) the upper limit. Usually, \( a < b \), but this is not essential.

We will make frequent use of the notation

\[
[F(x)]_{a}^{b} = F(b) - F(a)
\]

Thus

\[
\int_{a}^{b} f(x) \, dx = [F(x)]_{a}^{b}
\]

Note that in evaluating a definite integral, we drop the constant from the antiderivative \( F(x) \) because this constant of integration cancels in the final answer:

\[
[F(x) + C]_{a}^{b} = (F(b) + C) - (F(a) + C) = F(b) - F(a) = [F(x)]_{a}^{b}
\]

Example 1 Evaluate the following definite integrals.

\[
(a) \int_{a}^{b} x^2 \, dx \quad (b) \int_{2}^{4} \frac{1}{t} \, dt \quad (c) \int_{0}^{1} e^x \, dx
\]

Solution

(a)

\[
\int x^2 \, dx = \frac{x^3}{3} + C
\]

\[
\int_{a}^{b} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{a}^{b} = \frac{b^3}{3} - \frac{a^3}{3}
\]

(b)

\[
\int \frac{1}{t} \, dt = \ln |t| + C
\]

\[
\int_{2}^{3} \frac{1}{t} \, dt = [\ln |t|]_{2}^{3} = \ln 3 - \ln 2 = \ln 1.5
\]
\[
\begin{align*}
(c) \quad \int e^x \, dx &= e^x + C \\
\int_0^1 e^x \, dx &= [e^x]_0^1 = e^1 - e^0 = e - 1
\end{align*}
\]

One of the most important uses of definite integral is to calculate the area that is bounded by certain curves. Let \( f(x) \) be some given function defined and continuous in an interval \([a, b]\) and taking positive values in that interval. Then the area \( A \) enclosed by the curve \( y = f(x) \), the \( x \)-axis and the vertical lines \( x = a \) and \( x = b \) is given by

\[
A = \int_a^b f(x) \, dx = F(b) - F(a).
\]

where \( F(x) \) is an antiderivative of \( f(x) \).

The above formula constitutes the so called **fundamental theorem of calculus**, perhaps the most remarkable theorem in the whole calculus. Its prof is beyond the scope of this unit.

**Example 2** Evaluate the area between the curve \( y = x^2 \) and the \( x \)-axis from \( x = 0 \) to \( x = 2 \).

**Solution**

\[
A = \int_0^2 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}
\]

Definite integrals also have many other uses. If \( C(x) \) is the total cost of producing \( x \) units of a certain commodity, then \( C'(x) \) represents the marginal cost function. We have, by the definition of the definite integrals,

\[
\int_a^b C'(x) \, dx = [C(x)]_a^b = C(b) - C(a)
\]
But $C(b) - C(a)$ stands for the change of total cost when the production level is changed from $a$ units to $b$ units. It follows that $\int_a^b C'(x)\,dx$ represents the same change in total cost.

Similarly, if $R'(x)$ is the marginal revenue, then the change in revenue when the sales level changes from $a$ units to $b$ units is given by $\int_a^b R'(x)\,dx$. There is a similar explanation for $\int_a^b P'(x)\,dx$ where $P'(x)$ represents the marginal profit.

**Example 3** The marginal cost function for a firm at the production level $x$ is given by

$$C'(x) = 23 - 0.01x$$

Find the increase in total cost when the production level is increased from 1000 to 2000 units.

**Solution** The increase in total cost is given by

$$\int_{1000}^{2000} C'(x)\,dx = \int_{1000}^{2000} (23 - 0.01x)\,dx$$

$$= \left[ 23x - 0.01\frac{x^2}{2} \right]_{1000}^{2000}$$

$$= 23000 - 0.005(4000000 - 1000000)$$

$$= 23000 - 15000$$

$$= 8000$$

The cost increase is thus $8000$.

**Profit Maximizing Over Time**

There exist certain business operations such as mining and oil extraction that turn out to be non-profitable after a certain period of time. In such operations, the marginal revenue $R'(t)$ can be very high at the beginning of the operation but can decrease as time passes because of depletion of the resource. That is, $R'(t)$ eventually becomes a decreasing function of time $t$. On the other hand, the cost rate (marginal cost) $C'(t)$ of the operation is small at the beginning but often increases as time passes because of increased maintenance, etc. In such operations, there comes a time when the cost rate of running the operation exceeds that of the revenue and the operation starts to lose money. The management of such an operation is faced with selecting a time at which to close it down that will result in the maximum profit being obtained.

Let $C(t)$, $R(t)$, and $P(t)$ denote the total cost, total revenue, and the total profit up to time $t$ (measured from the start of the operation), respectively. Then

$$P(t) = R(t) - C(t) \text{ and } P'(t) = R'(t) - C'(t)$$
The maximum total profit occurs when

\[ P'(t) = 0, \text{ or } R'(t) = C'(t) \]

Therefore, the operation should run until the time \( t_1 \), at which

\[ R'(t_1) = C'(t_1) \quad (\text{assuming } P''(t_1) < 0, \text{ i.e., } R''(t_1) < C''(t_1)) \]

**Example 4** The cost and the revenue rates for a certain mining operation are given by

\[ C'(t) = 5 + 2t^\frac{2}{3} \quad \text{and} \quad R'(t) = 17 - t^\frac{2}{3}, \]

where \( C \) and \( R \) are measured in millions of dollars and \( t \) is measured in years. Determine how long the operation should continue and find the total profit that can be earned during this period.

**Solution** Let \( P'(t) = R'(t) - C'(t) = 0 \). We obtain

\[
\begin{align*}
5 + 2t^\frac{2}{3} &= 17 - t^\frac{2}{3} \\
3t^\frac{2}{3} &= 17 - 5 = 12 \\
t^\frac{2}{3} &= 4 \\
t &= 4^\frac{3}{2} = 8 \\
P''(t) &= (17 - t^\frac{2}{3})' - (5 + 2t^\frac{2}{3})' = -\frac{2}{3}t^{-\frac{1}{3}} - \frac{4}{3}t^{-\frac{1}{3}} \\
&= -\frac{2}{3}t^{-\frac{1}{3}}.
\end{align*}
\]

When \( t = 8 \), \( P''(8) = -2(8)^{-\frac{1}{3}} < 0 \). Hence \( P \) has a maximum at \( t = 8 \). Thus the operation should continue for 8 years. The total profit earned during this period of 8 years is given by

\[
\int_0^8 P'(t)dt = \int_0^8 \left[(17 - t^\frac{2}{3}) - (5 + 2t^\frac{2}{3})\right] dt
\]

\[
\begin{align*}
&= \int_0^8 (12 - 3t^\frac{2}{3}) dt \\
&= \left[12t - 3\frac{2}{3}t^\frac{2}{3} + 1\right]_0^8 \\
&= 12 \times 8 - \frac{9}{5}(8)^\frac{5}{3} = 38.2 \text{ (millions of dollars)}. \]
\]