Use Maxima to answer the following questions taken from past MATH101 and MATH102 examination papers.

1. For each answer include your working in Maxima. Where appropriate express the final result using \texttt{LATEX}.

2. The questions do not carry equal marks.

3. You should complete Assignment 3 before attempting this assignment.

4. This assignment will be the subject of Practical 4.

**Question 1**

[7/2 marks]

(a) For \( z = 1 - i \) express each of the following complex numbers in the form \( x + iy \), with \( x \) and \( y \) real.

\[
\begin{align*}
(i) & \quad 3z - i \\
(ii) & \quad \frac{z - i}{z + i} \\
(iii) & \quad 2z + i \\
(iv) & \quad \left| \frac{z}{z + i} \right| \\
(v) & \quad \overline{z^2}.
\end{align*}
\]

(b) Find, over \( \mathbb{C} \), all solutions of the equation \( z^4 = i \), i.e. find (over the complex numbers) all possible values of \( i^{\frac{1}{4}} \).

**Question 2**

[3/2 marks]

(a) Discuss the behaviour of each of the sequences as \( n \to \infty \).

\[
\begin{align*}
(i) & \quad \left( \frac{7}{8} \right)^n \\
(ii) & \quad \frac{1 - n}{4n + 1} \\
(iii) & \quad \frac{\cos(n)}{n}.
\end{align*}
\]

(b) Determine if the following infinite series are convergent or divergent.

\[
\begin{align*}
(i) & \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 2n}} \\
(ii) & \quad \sum_{n=1}^{\infty} \frac{4^n + 1}{6^n} \\
(iii) & \quad \sum_{n=1}^{\infty} \frac{2^n}{n!}.
\end{align*}
\]
Question 3

[3/2 marks]

(a) Differentiate the following functions:

(i) \( f(x) = \sqrt{1 + e^{3x^2}} \)  
(ii) \( f(x) = x \ln(1 + x^2) \).

(b) Find \( \frac{dy}{dx} \) for the following implicitly defined function,

\[ \tan(xy) + xy + y^2 = 0. \]

Question 4

[3/2 marks]

Consider the function, \( f : \mathbb{R} \rightarrow \mathbb{R} \)

\[ f(x) = x^3 + 3x^2 - 9x + 6. \]

(a) Determine the intervals on which \( f \) is (i) increasing or decreasing and (ii) concave up or concave down.

(b) Find all relative maxima and minima of the function \( f \). What are the absolute maxima and minima for \( f \) on the interval \([-3, 2]\)?

(c) Sketch the graph of \( f \) on the interval \([-3, 2]\).

Question 5

[6/2 marks]

(a) Solve the following linear system,

\[
\begin{align*}
 x - 2y + z &= 9 \\
 x + y + z &= 6 \\
-3x + 2y + z &= -7 
\end{align*}
\]

(b) For

\[
A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}
\]

state whether the following products and (or) sums are defined; calculate those which are defined.

(i) \( AB \)  
(ii) \( A - C \)  
(iii) \( BC \)
(c) If \[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & j
\end{vmatrix} = 5,
\]
evaluate the following determinants.

(i) \[
\begin{vmatrix}
  a & b & 5c \\
  d & e & 5f \\
  g & h & 5j
\end{vmatrix}
\]

(ii) \[
\begin{vmatrix}
  a - d & b - e & c - f \\
  d & e & f \\
  g & h & j
\end{vmatrix}
\]

Question 6

[4/2 marks]

Let \( P_1, P_2, P_3 \) and \( P_4 \) be the following four points in \( \mathbb{R}^3 \), \( P_1(1, 0, 0), P_2(0, 0, 1), \) and \( P_3(1, 1, 1) \).

(a) Write down the vectors \( \vec{P_1P_2} \) and \( \vec{P_1P_3} \) in terms of the standard unit vectors \( i, j \) and \( k \).

(b) Find the orthogonal projection of \( \vec{P_1P_2} \) onto \( \vec{P_1P_3} \).

(c) Find the area of the triangle formed by \( P_1, P_2 \) and \( P_3 \)

(d) Find the volume of the parallelepiped with sides given by the three vectors \( \vec{P_1P_2}, \vec{P_1P_3} \) and \( \vec{P_1P_4} \).

Question 7

[6/2 marks]

Consider a rectangle inscribed in a semicircle of radius \( a \), i.e. two of the vertices of the rectangle are on the circular arc and one side of the rectangle lies on part of the diameter.

(a) What are the dimensions of the rectangle of largest area which can be inscribed in a semicircle as above?

(b) What are the dimensions of the rectangle with largest perimeter which can be inscribed in the semicircle?

Question 8

[2/2 marks]

Prove that if \( \mathbf{u} \) and \( \mathbf{v} \) are two vectors in \( \mathbb{R}^3 \) then

\[
|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2.
\]
Question 9

[4/2 marks]

(a) Find
\[ \int \frac{2x + 1}{\sqrt{x^2 + x - 3}} \, dx \]

(b) Compute
\[ \int_{0}^{1} \tan^{-1}(x) \, dx \]

(c) Calculate
\[ \int \frac{dx}{x^3 - 3x + 2} \]

(d) Evaluate the improper integral
\[ \int_{1}^{\infty} x^{-1.01} \, dx. \]