A graph consists of a set of vertices and a set of edges, where each edge joins a pair of vertices.
The set of vertices of a graph $G$ is denoted by $V(G)$, the set of edges is denoted by $E(G)$.

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

Notes:
1. A graph may any number of vertices and any number of edges.
2. An edge may join a vertex to itself.
3. More than one edge may join a pair of vertices.
4. A vertex need not be attached to any edges.
5. An edge does not have a direction.
To describe a graph $G$ completely we need to give:

1. The set of vertices $V(G)$.
2. The set of edges $E(G)$.
3. The edge-endpoint function which associates each edge with its endpoints, i.e. a pair of vertices.

For our example the edge-endpoint function is given by:

<table>
<thead>
<tr>
<th>edge</th>
<th>endpoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>${v_1}$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>${v_1, v_2}$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>${v_2, v_3}$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>${v_2, v_3}$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>${v_2, v_3}$</td>
</tr>
</tbody>
</table>
Some Terminology

1. An edge which joins a vertex to itself is called a loop, e.g. the edge $e_1$ in our example.

2. Edges which join the same pair of vertices are said to be parallel. In our example the edges $e_3, e_4, e_5$ are parallel.

3. Two vertices joined by an edge are said to be adjacent. In our example the vertices $v_1$ and $v_2$ are adjacent, but the vertices $v_1$ and $v_3$ are not adjacent.

Directed Graphs

A directed graph or digraph differs from a graph in that each edge has a direction.
To describe a directed graph $D$ we need to give:

1. The set of vertices $V(D)$.
2. The set of edges $E(D)$.
3. The edge-endpoint function which associates each edge with its endpoints, in this case an ordered pair of vertices.

The edge associated with an ordered pair of vertices $(v_i, v_j)$ is directed from $v_i$ to $v_j$.

For our example digraph the edge-endpoint function is given by:

<table>
<thead>
<tr>
<th>edge</th>
<th>endpoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$(v_1, v_1)$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$(v_1, v_2)$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$(v_2, v_3)$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>$(v_3, v_2)$</td>
</tr>
</tbody>
</table>
Types of Graphs

A **simple graph** is a graph without any loops or parallel edges.  
Example:

A **complete graph** is a simple graph where each pair of vertices is joined by an edge.  
Example:
The complete graph on \( n \) vertices is denoted by \( K_n \).

The first few are:

\[
\begin{align*}
K_1 & \quad K_2 & \quad K_3 & \quad K_4
\end{align*}
\]

A **complete bipartite graph** is a simple graph where the vertices are divided into two sets, say \( V_1 \) and \( V_2 \), so that there is an edge from each vertex in \( V_1 \) to each vertex in \( V_2 \), but no edges between any vertices in \( V_1 \) and no edges between any vertices in \( V_2 \).

Example:
The complete bipartite graph on $m$ and $n$ vertices is denoted by $K_{m,n}$.
Here are some:

\[ K_{1,2} \quad K_{2,2} \quad K_{3,3} \]

A graph $H$ is a subgraph of a graph $G$ if
1. Every vertex of $H$ is a vertex of $G$.
2. Every edge of $H$ is a edge of $G$.
3. Every edge of $H$ has the same endpoints as in $G$.

**Example:** The graph
Has subgraphs:

\[ H_1 \]

\[ H_2 \]

There are many other possibilities.

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**Degree of a Vertex**

The **degree** of a vertex \( v \) of a graph \( G \), denoted by \( \delta(v) \) or \( \deg(v) \), is the number of edges incident on \( v \), with a loop counted twice.

The **total degree** of a graph \( G \) is the sum of the degrees of all the vertices of \( G \).
Example:

\[
\begin{align*}
\text{deg}(v_1) &= 3 \\
\text{deg}(v_2) &= 4 \\
\text{deg}(v_3) &= 3 \\
\text{deg}(v_4) &= 0 \\
\text{deg}(v_5) &= 0 \\
\text{Total degree} &= 10
\end{align*}
\]

**Theorem:** For any graph $G$ the total degree of $G$ is equal to twice the number of edges of $G$.

**Proof:** Each edge (including loops) contributes two to the total degree of the graph. Thus the total degree is equal to twice the number of edges.

**Example:** Our example graph has five edges and total degree 10.
Example: The complete bipartite graph $K_{2,3}$

Has 3 vertices with degree 2 and 2 vertices with degree 3, giving total degree 12.

There are 6 edges.