Big $O$ Notation

Recall the definition:

Let $f(x)$ and $g(x)$ be two functions. We say that $f(x)$ is $O(g(x))$ or $f(x) = O(g(x))$ if there are numbers $M$ and $C$ such that

$$|f(x)| \leq C|g(x)| \quad \text{for } x \geq M$$

Triangle Inequality:

$$|x \pm y| \leq |x| + |y|$$
Example

Let

\[ f(x) = \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \]

We will show

\[ f(x) = O(\sqrt{x}) \]

We need to show that

\[ |f(x)| \leq C\sqrt{x} \]

holds for some value \( C \), for \( x \) greater than some value \( M \).

\[ |f(x)| = \left| \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \right| \]

\[ = \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \]

\[ \leq \frac{3\sqrt{x}(2|x| + 5)}{|x|} \quad \text{(since } |x| + 1 > |x|) \]

\[ \leq \frac{3\sqrt{x}(2|x| + |x|)}{|x|} \quad \text{(triangle inequality)} \]

\[ \leq \frac{3\sqrt{x}(|2x| + |x|)}{|x|} \quad \text{for } x \geq 5 \]
Thus we have

\[ |f(x)| \leq 9\sqrt{x} \quad \text{for } x \geq 5 \]

and so \( f(x) = O(\sqrt{x}) \).

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**Example**

We will show

\[ n^3 = O(n^5) \]

We need to show that

\[ |n^3| \leq C|n^5| \]

holds for some value \( C \), for \( n \) greater than some value \( M \).
This one is easy

\[|n^3| \leq |n^2||n^3| \quad \text{(for } n \geq 1, \text{ since } |n^2| \geq 1)\]
\[= |n^5|\]

More generally we have

\[n^a = O(n^b) \quad \text{for } a \leq b\]

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Some Properties of \(O\)

1. \(f(x) = O(f(x))\).

2. \(c \cdot f(x) = O(f(x))\) for any constant \(c\).

3. If \(f(x) = O(g(x))\) then \(c \cdot f(x) = O(g(x))\) for any constant \(c\).

4. If \(f(x) = O(g(x))\) and \(g(x) = O(h(x))\) then \(f(x) = O(h(x))\)
Proof of 4:

\[ f(x) = O(g(x)) \] says there are constants \( C \) and \( M \) such that

\[ |f(x)| \leq C|g(x)| \quad \text{for } x \geq M \]

\[ g(x) = O(h(x)) \] says there are constants \( D \) and \( N \) such that

\[ |g(x)| \leq D|h(x)| \quad \text{for } x \geq N \]

Thus

\[ |f(x)| \leq C|g(x)| \leq CD|h(x)| \quad \text{for } x \geq \max(M, N) \]

**Big \( \Theta \) Notation**

Here is the mathematical definition:

*Let \( f(x) \) and \( g(x) \) be two functions. We say that \( f(x) \) is \( \Theta(g(x)) \) or \( f(x) = \Theta(g(x)) \) if there are numbers \( C, D \) and \( M \) such that

\[ D|g(x)| \leq |f(x)| \leq C|g(x)| \quad \text{for } x \geq M \]

In words: the absolute value of \( f(x) \) is “sandwiched” between multiples of the absolute value of \( g(x) \) for large enough \( x \).
Example

Let $f(n) = 2n - 1$. We will show $f(n) = \Theta(n)$.

We need to show that both

$$|2n - 1| \leq C|n|$$

and

$$D|n| \leq |2n - 1|$$

hold for $n$ greater than some value $M$.

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We have already shown

$$|2n - 1| \leq 3|n| \quad \text{for } n \geq 1$$

in the previous lecture.

Now we need to show:

$$D|n| \leq |2n - 1|$$

Equivalently:

$$|2n - 1| \geq D|n|$$
Another form of the **Triangle Inequality**:

\[ |x \pm y| \geq |x| - |y| \]

\[ |2n - 1| \geq |2n| - 1 \] (triangle inequality)

\[ \geq |2n| - |n| \] (for \( n \geq 1 \))

\[ = |n| \]

so

\[ |n| \leq |2n - 1| \] for \( n \geq 1 \)

Now we have

\[ |2n - 1| \leq 3|n| \] for \( n \geq 1 \)

and

\[ |n| \leq |2n - 1| \] for \( n \geq 1 \)

i.e.

\[ |n| \leq |2n - 1| \leq 3|n| \] for \( n \geq 1 \)

Thus

\[ 2n - 1 = \Theta(n) \]
**Example**

Let $f(n) = \frac{1}{2}n(n + 1)$. We will show $f(n) = \Theta(n^2)$.

We have already shown

$$\left| \frac{1}{2}n(n + 1) \right| \leq |n^2| \text{ for } n \geq 1$$

We need to show

$$D|n^2| \leq \left| \frac{1}{2}n(n + 1) \right|$$

for some $D$ and for $n \geq N$ for some $N$.

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Again turn the inequality around and show

$$\left| \frac{1}{2}n(n + 1) \right| \geq D|n^2|$$

Now

$$\left| \frac{1}{2}n(n + 1) \right| = \frac{1}{2}|n||n + 1|$$

$$\geq \frac{1}{2}|n||n| \text{ for } n \geq 0$$

$$\geq \frac{1}{2}|n^2|$$
Now we have
\[
\frac{1}{2} n^2 \leq \left| \frac{1}{2} n(n + 1) \right| \leq |n^2|
\]
for \( n \geq 1 \).

Thus
\[
\frac{1}{2} n(n + 1) = \Theta(n^2)
\]

**Example**

Let
\[
f(x) = \frac{3\sqrt{x}(2x + 5)}{|x| + 1}
\]

We will show
\[
f(x) = \Theta(\sqrt{x})
\]

Again, we have already done half of the work:
\[
\left| \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \right| \leq 9\sqrt{x} \quad \text{for } x \geq 5
\]
We need to show
\[
\left| \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \right| \geq D\sqrt{x} \quad \text{for } x \geq N
\]

We have
\[
\left| \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \right| = 3\frac{|\sqrt{x}|2x + 5}{|x| + 1}
\geq 3\frac{|\sqrt{x}|2x + 5}{|x| + |x|} \quad (\text{for } x \geq 1)
\]
\[
\quad \quad \text{since } (|x| + |x|) \geq |x| + 1
\]
\[
= 3\frac{|\sqrt{x}|2x + 5}{|2x|}
\]

\[
\geq 3\frac{|\sqrt{x}|2x}{|2x|}
\]
\[
\quad \quad \text{since } |2x + 5| \geq |2x|
\]
\[
= 3|\sqrt{x}|
\]

Thus
\[
3\sqrt{x} \leq \left| \frac{3\sqrt{x}(2x + 5)}{|x| + 1} \right| \leq 9\sqrt{x}
\]
for \( x \geq 5 \).