Evaluating Polynomials

Problem:

Evaluate a polynomial of degree $n$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

at a point $x$. 
**Standard Method**

1. Evaluate the powers \( x, x^2, \ldots x^n \).
   
   We do this by starting with \( x \) and successively multiplying by \( x \) to obtain \( x^2 \), then \( x^3 \), \ldots, and finally \( x^n \),

2. Form the products \( a_0, a_1x, a_2x^2, \ldots a_nx^n \).

3. Add them together:
   
   \[ p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \]

**Operation Count**

1. Evaluating the powers \( x, x^2, \ldots x^n \) requires \( n - 1 \) multiplications — one for each of \( x^2, \ldots x^n \).

2. Evaluating the products \( a_0, a_1x, a_2x^2, \ldots a_nx^n \) requires \( n \) multiplications — one for each of \( a_1x, \ldots a_nx^n \).

3. Evaluating the sum \( a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \) requires \( n \) additions.

In total: \( 2n - 1 \) multiplications and \( n \) additions.
Horner’s Method

Rewrite the polynomial

\[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

as

\[ p(x) = x[x[\cdots x(x(a_n x + a_{n-1}) + a_{n-2}) + \cdots] + a_1] + a_0 \]

For example:

\[ 5x^4 + 3x^3 + 7x^2 + 2x + 4 = x(x(x(5x^2 + 3) + 2) + 4 \]

Evaluate

\[ p(x) = x[x[\cdots x(x(a_n x + a_{n-1}) + a_{n-2}) + \cdots] + a_1] + a_0 \]

as indicated by the brackets:

1. Multiply \( a_n \) by \( x \).
2. Add \( a_{n-1} \) to the result.
3. Multiply the result by \( x \).
4. Add \( a_{n-2} \) to the result.
5. Multiply the result by \( x \), etc
Operation Count

1. The steps of Horner’s methods alternate between multiplications and additions.
2. There are the same number of multiplications and additions.
3. The terms to be added run from $a_{n-1}$ down to $a_0$ — $n$ terms in total.

In total: $n$ multiplications and $n$ additions.

Example

We will use evaluate

$$5x^4 + 3x^3 + 7x^2 + 2x + 4 = x(x(x(5x + 3) + 7) + 2) + 4$$

at $x = 8$.

Result = 22484.
Standard Method

1. Start with $x = 8$ and evaluate by successive multiplication the powers $8^2 = 8 \cdot 8 = 64, 8^3 = 8 \cdot 64 = 512, 8^4 = 8 \cdot 512 = 4096$ (3 multiplications).

2. Form the products
   $2 \cdot 8 = 16, 7 \cdot 64 = 448, 3 \cdot 512 = 1536, 5 \cdot 4096 = 20480$ (4 multiplications).

3. Add them together: $4 + 16 + 448 + 1536 + 20480 = 22484$ (4 additions).

Total: 7 multiplications, 4 additions.

Horner’s Method

1. Multiply 5 by 8 add 3, result = 43.

2. Multiply 43 by 8 add 7, result = 351.

3. Multiply 351 by 8 add 2, result = 2810.

4. Multiply 2810 by 8 add 4, result = 22484.

Each step consists of 1 multiplication and 1 addition.

Total: 4 multiplications, 4 additions.
Summary

To evaluate a polynomial of degree $n$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

at a point $x$ requires

**Standard Method:** $(2n - 1)$ multiplications, $n$ additions, roughly $3n$ operations.

**Horner’s Method:** $n$ multiplications, $n$ additions, roughly $2n$ operations.

Horner’s method requires about $2/3$ of the number of operations as the standard method.

Both Horner’s method and the standard method have the property that the number of arithmetic operations needed to evaluate a polynomial of degree $n$ is (roughly) proportional to $n$.

The standard method requires about $3n$ operations.

Horner’s method requires $2n$ operations.
This means that evaluating a polynomial of degree 100 will take about 10 times as many operations as evaluating a polynomial of degree 10 (using the same method for both).

We say that this methods are $O(n)$ ‘big oh of $n$’, roughly meaning that, at least for large $n$, the number of operations is proportional to $n$.

**Big $O$ Notation**

Big $O$ notation gives a precise for comparing the efficiency in space and time of algorithms.

Here is the mathematical definition:

Let $f(x)$ and $g(x)$ be two functions. We say that $f(x)$ is $O(g(x))$ or $f(x) = O(g(x))$ if there are numbers $M$ and $C$ such that

$$|f(x)| \leq C|g(x)| \quad \text{for } x \geq M$$

In words: the absolute value of $f(x)$ is less than a constant multiple ($C$) of the absolute value of $g(x)$ for large enough $x \ (\geq M)$. 
Example

Let \( f(n) = 2n - 1 \). We will show \( f(n) = O(n) \).

We need to show that
\[
|2n - 1| \leq Cn
\]
holds for some value \( C \), for \( n \) greater than some value \( M \). Values for \( C \) and \( M \) have to be found (the particular values themselves are not very important).

Triangle Inequality:
\[
|x \pm y| \leq |x| + |y|
\]

\[
|2n - 1| \leq |2n| + |1| \quad \text{(triangle inequality)}
\]
\[
\leq |2n| + |n| \quad \text{(for } n \geq 1)\]
\[
= |3n| = 3|n|
\]

Thus \( f(n) = 2n - 1 \) satisfies:
\[
|f(n)| \leq 3|n| \quad \text{for } n \geq 1
\]
i.e. \( f(n) = O(n) \).
Example

Let \( f(n) = \frac{1}{2}n(n + 1) \). We will show \( f(n) = O(n^2) \).

We need to show that

\[
\left| \frac{1}{2}n(n + 1) \right| \leq C|n^2|
\]

holds for some value \( C \), for \( n \) greater than some value \( M \).

\[
\left| \frac{1}{2}n(n + 1) \right| = \frac{1}{2}|n||(n + 1)|
\]

\[
\leq \frac{1}{2}|n|(|n| + |1|) \quad \text{(triangle inequality)}
\]

\[
\leq \frac{1}{2}|n|(|n| + |n|) \quad \text{(for } n \geq 1\text{)}
\]

\[
\leq \frac{1}{2}|n||2n|
\]

\[
= |n^2|
\]
Thus $f(n) = \frac{1}{2} n(n + 1)$ satisfies

$$|f(n)| \leq |n^2| \quad \text{for } n \geq 1$$

i.e. $f(n) = O(n^2)$. 