Sets

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Reading: Lecture Notes §1, Epp §5.1, §5.2

Sets and Elements

1. A set is a collection of elements.

2. A set is completely determined by its elements.

3. The notation
   \[ x \in X \]
   is used to denote that the element \( x \) is in the set \( X \).

4. The notation
   \[ x \notin X \]
   is used to denote that the element \( x \) is not in the set \( X \).
Notations for Sets

1. The elements of set can be listed:

\[ \{1, 97, -1234\} \]

denotes the set containing just the three numbers 1, 97 and -1234.

\[ \{1, 3, 5, 7, \ldots\} \]

denotes the set of odd positive integers.

2. Some common sets:
   (a) \( \mathbb{R} \) – the set of real numbers.
   (b) \( \mathbb{Z} \) – the set of integers.
   (c) \( \mathbb{N} \) – the set of natural numbers, i.e. \( \{1, 2, 3, 4, \ldots\} \).

3. A set can be defined by its properties:

\[ \{n \mid n \text{ is an odd positive integer}\} \]

is another way of denoting the set \( \{1, 3, 5, 7, \ldots\} \).

\[ \mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\} \]

is another way of defining the set of natural numbers.
Relations Between Sets

1. Two sets $A$ and $B$ are equal, written $A = B$ if they have exactly the same elements.

2. A set $A$ is a subset of a set $B$, written $A \subseteq B$ if every element of $A$ is an element of $B$.

According to these definitions

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

Operations on Sets

1. The union of two sets $A$ and $B$ is the set of elements which are in either $A$ or $B$:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

2. The intersection of two sets $A$ and $B$ is the set of elements which are in both $A$ and $B$:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

3. The relative complement of set $A$ in set $B$ is the set of elements which are in $B$ but not in $A$:

$$B \setminus A = \{x \in B \mid x \notin A\}$$
Empty Set, Power Sets, Cartesian Products

1. The **empty** set, denoted by $\emptyset$, is the set which has no elements.
2. The **power set** of a set $A$, denoted by $\mathcal{P}(A)$, is the set of all subsets of $A$:
   \[ \mathcal{P}(A) = \{ X \mid X \subseteq A \} \]
   For example, if $A = \{0, 1\}$ then
   \[ \mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \]
   For every set $A$
   \[ \emptyset \subseteq \mathcal{P}(A) \text{ and } A \subseteq \mathcal{P}(A) \]

3. The **Cartesian product** of two sets $A$ and $B$, denoted by $A \times B$, is the set of **ordered pairs** elements of $A$ and $B$
   \[ A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\} \]
   For example, if $A = \{0, 1\}$ and $B = \{x, y, z\}$ then
   \[ A \times B = \{(0, x), (0, y), (0, z), (1, x), (1, y), (1, z)\} \]
Partitions

1. Sets $A$ and $B$ are **disjoint** if they have no common elements:
   \[ A \cap B = \emptyset \]

2. Sets $A_1, A_2, \ldots, A_n$ are **mutually** disjoint if each pair of distinct sets is disjoint:
   \[ A_i \cap A_j = \emptyset \text{ whenever } i \neq j \]

3. A **partition** of a set $A$ is a collection of non-empty subsets of $A$, $A_1, A_2, \ldots, A_n$ with the properties that
   (a) $A_1, A_2, \ldots, A_n$ are mutually disjoint.
   (b) $A = A_1 \cup A_2 \cup \cdots \cup A_n$.

Subset Relations

1. (a) $A \cap B \subseteq A$
   (b) $A \cap B \subseteq B$

2. (a) $A \subseteq A \cup B$
   (b) $B \subseteq A \cup B$

3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. 
Set Identities

1. Commutative Laws:
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$

2. Associative Laws:
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$

3. Distributive Laws:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. Identity Laws:
   (a) $A \cup \emptyset = A$
   (b) $A \cap \emptyset = \emptyset$

5. Idempotent Laws:
   (a) $A \cup A = A$
   (b) $A \cap A = A$

6. Absorption Laws:
   (a) $A \cup (A \cap B) = A$
   (b) $A \cap (A \cup B) = A$
Complements

1. Often the sets we are interested in are all subsets of some universal set $U$. For any set $A$ the complement of $A$, denoted by $A'$ (other notations are $\bar{A}$ and $A^c$), is the set of elements which are in $U$ but not in $A$:

$$A' = U - A = \{x \in U \mid x \notin A\}$$

2. Complement Laws:
   (a) $A \cup A' = U$
   (b) $A \cap A' = \emptyset$


4. De Morgan’s Laws:
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$